Online Appendix for "Innovation, Reallocation and Growth"

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Appendix A: Proofs and Derivations

Proof of Lemma 1. Consider the low-type firms and conjecture $\tilde{V}_l\left(\hat{\mathcal{Q}}\right) = \sum_{\hat{q}\in\hat{\mathcal{Q}}} \Upsilon^l\left(\hat{q}\right)$:

$$r\sum_{\hat{q}\in\hat{\mathcal{Q}}} \Upsilon^l\left(\hat{q}\right) = \sum_{\hat{q}\in\hat{\mathcal{Q}}} \max\left\{0, \max_{x\geq 0}\left[\begin{array}{c} \tilde{\pi}\left(\hat{q}_j\right) - \tilde{w}^s\phi^s - \tilde{w}^s G\left(x,\theta^L\right) + \frac{\partial \Upsilon^l(\hat{q})}{\partial \hat{q}}\frac{\partial \hat{q}}{\partial w^u}\frac{\partial w^u}{\partial t} \\ +x\mathbb{E}\Upsilon^l\left(\hat{q}+\lambda\bar{\hat{q}}\right) - \left(\tau+\varphi\right)\Upsilon^l\left(\hat{q}\right) \end{array}\right]\right\},$$

which implies

$$r\Upsilon^l\left(\hat{q}\right) = \max\left\{0, \left\{\begin{array}{c} \tilde{\pi}\left(\hat{q}\right) - \tilde{w}^s\phi^s + \frac{\partial \Upsilon^l(\hat{q})}{\partial \hat{q}}\frac{\partial \hat{q}}{\partial w^u}\frac{\partial w^u}{\partial t} - \left(\tau+\varphi\right)\Upsilon^l\left(\hat{q}\right) \\ +\max_{x\geq 0}\left[x\mathbb{E}\Upsilon^l\left(\hat{q}+\lambda\bar{\hat{q}}\right) - \tilde{w}^s G\left(x,\theta^L\right)\right] \end{array}\right\}\right\},$$

where we also use the fact that a firm can choose not to operate an individual product line.

Next consider the high-type firms and conjecture $\tilde{V}_h\left(\hat{\mathcal{Q}}\right) = \sum_{\hat{q}\in\hat{\mathcal{Q}}} \Upsilon^h\left(\hat{q}\right)$:

$$r\sum_{\hat{q}\in\hat{\mathcal{Q}}} \Upsilon^h\left(\hat{q}\right) = \sum_{\hat{q}\in\hat{\mathcal{Q}}} \max\left\{0, \max_{x\geq 0}\left[\begin{array}{c} \tilde{\pi}\left(\hat{q}\right) - \tilde{w}^s\phi^s - \tilde{w}^s G\left(x,\theta^H\right) + \frac{\partial \Upsilon^h(\hat{q})}{\partial \hat{q}}\frac{\partial \hat{q}}{\partial w^u}\frac{\partial w^u}{\partial t} \\ +x\mathbb{E}\Upsilon^h\left(\hat{q}+\lambda\bar{\hat{q}}\right) \\ -\left(\tau+\varphi\right)\Upsilon^h\left(\hat{q}\right) + \nu\left[\mathbb{I}_{\hat{q}>\hat{q}_{l,\min}}\cdot\Upsilon^l\left(\hat{q}\right) - \Upsilon^h\left(\hat{q}\right)\right] \end{array}\right]\right\},$$

which similarly implies

$$r\Upsilon^h\left(\hat{q}\right) = \max\left\{0, \max_{x\geq 0}\left[\begin{array}{c} \tilde{\pi}\left(\hat{q}\right) - \tilde{w}^s\phi^s - \tilde{w}^s G\left(x,\theta^H\right) + \frac{\partial \Upsilon^h(\hat{q})}{\partial \hat{q}}\frac{\partial \hat{q}}{\partial w^u}\frac{\partial w^u}{\partial t} + \\ x\mathbb{E}\Upsilon^h\left(\hat{q}+\lambda\bar{\hat{q}}\right) \\ -\left(\tau+\varphi\right)\Upsilon^h\left(\hat{q}\right) + \nu\left[\mathbb{I}_{\hat{q}>\hat{q}_{l,\min}}\cdot\Upsilon^l\left(\hat{q}\right) - \Upsilon^h\left(\hat{q}\right)\right] \end{array}\right]\right\}.$$

Monotonicity follows from the fact that the per-period return function is increasing in $\hat{q}$.
■

Proof of Proposition 1. First note that $\tilde{\pi}(q) = \left(\frac{\varepsilon-1}{\varepsilon}\right)^{\varepsilon} \frac{1}{\varepsilon-1}\hat{q}^{\varepsilon-1} = \Pi\hat{q}^{\varepsilon-1}$. Then, defining $\Psi \equiv r+\tau+\varphi$, equation (17) can be written as the following linear differential equation

$$\Psi Y^{l}(\hat{q}) + g\hat{q}\frac{\partial Y^{l}(\hat{q})}{\partial \hat{q}} = \Pi\hat{q}^{\varepsilon-1} + \Omega^{l} - \tilde{w}^{s}\phi \text{ if } \hat{q} > \hat{q}_{l,\min}$$

or

$$\xi_1\hat{q}^{-1}Y^{l}(\hat{q}) + \frac{\partial Y^{l}(\hat{q})}{\partial \hat{q}} = \xi_2\hat{q}^{\varepsilon-2} - \xi_3\hat{q}^{-1}, \tag{A-1}$$

where $\xi_1 \equiv \frac{\Psi}{g}$, $\xi_2 \equiv \frac{\Pi}{g}$ and $\xi_3 \equiv \frac{\tilde{w}^{s}\phi-\Omega^{l}}{g}$. Then the solution to (A-1) can be written as

$$Y^{l}(\hat{q}) = \hat{q}^{-\xi_1}\left(\int\left[\xi_2 t^{\xi_1+\varepsilon-2} - \xi_3 t^{\xi_1-1}\right]dt + D\right) = \frac{\xi_2\hat{q}^{\varepsilon-1}}{\xi_1+\varepsilon-1} - \frac{\xi_3}{\xi_1} + D\hat{q}^{-\xi_1}.$$

Imposing the boundary condition $Y^{l}(\hat{q}_{l,\min}) = 0$, we can solve out for the constant of integration D, obtaining

$$\begin{aligned} Y^{l}(\hat{q}) &= \frac{\xi_2\hat{q}^{\varepsilon-1}}{\xi_1+\varepsilon-1} - \frac{\xi_3}{\xi_1} + \left(\frac{\xi_3\hat{q}_{l,\min}^{\xi_1}}{\xi_1} - \frac{\xi_2\hat{q}_{l,\min}^{\xi_1+\varepsilon-1}}{\xi_1+\varepsilon-1}\right)\hat{q}^{-\xi_1} \qquad \text{(A-2)} \\ &= \frac{\Pi\hat{q}^{\varepsilon-1}}{\Psi+(\varepsilon-1)g}\left(1-\left(\frac{\hat{q}_{l,\min}}{\hat{q}}\right)^{\frac{\Psi}{g}+\varepsilon-1}\right) + \frac{\Omega^{l}-\tilde{w}^{s}\phi}{\Psi}\left(1-\left(\frac{\hat{q}_{l,\min}}{\hat{q}}\right)^{\frac{\Psi}{g}}\right). \end{aligned}$$

We next provide the derivation of the value for a high-type product line. Let us rewrite the expression in (A-2) as

$$Y^{l}(\hat{q}) = \xi_4\hat{q}^{\varepsilon-1} + \xi_5\hat{q}^{-\frac{\Psi}{g}} - \xi_6,$$

where

$$\xi_4 \equiv \frac{\Pi}{\Psi+(\varepsilon-1)g},\ \xi_5 = \frac{\left(\tilde{w}^{s}\phi-\Omega^{l}\right)\hat{q}_{l,\min}^{\frac{\Psi}{g}}}{\Psi} - \frac{\Pi\hat{q}_{l,\min}^{\frac{\Psi}{g}+\varepsilon-1}}{\Psi+g(\varepsilon-1)},\ \text{and } \xi_6 = \frac{\tilde{w}^{s}\phi-\Omega^{l}}{\Psi}.$$

Recall the value of a product line of a high-type firm

$$\begin{aligned} (\Psi+\nu)Y^{h}(\hat{q}) + \frac{\partial Y^{h}(\hat{q})}{\partial \hat{q}}g\hat{q} &= \Pi\hat{q}^{\varepsilon-1} + \Omega^{h} - \tilde{w}^{s}\phi + \nu\left(\xi_4\hat{q}^{\varepsilon-1} + \xi_5\hat{q}^{-\frac{\Psi}{g}} - \xi_6\right) \text{ for } \hat{q} \geq \hat{q}_{l,\min} \\ (\Psi+\nu)Y^{h}(\hat{q}) + \frac{\partial Y^{h}(\hat{q})}{\partial \hat{q}}g\hat{q} &= \Pi\hat{q}^{\varepsilon-1} + \Omega^{h} - \tilde{w}^{s}\phi \text{ for } \hat{q}_{l,\min} > \hat{q} \geq \hat{q}_{h,\min}, \end{aligned}$$

which can be rewritten as

$$K_1 Y^{h}(\hat{q})\hat{q}^{-1} + \frac{\partial Y^{h}(\hat{q})}{\partial \hat{q}} = K_2\hat{q}^{\varepsilon-2} + K_3\hat{q}^{-\frac{\Psi+g}{g}} - K_4\hat{q}^{-1},$$

where

$$K_1 \equiv \frac{\Psi+\nu}{g},\ K_2 \equiv \frac{\Pi+\nu\xi_4}{g},\ K_3 \equiv \frac{\nu\xi_5}{g} \text{ and } K_4 \equiv \frac{\nu\xi_6+\tilde{w}^s\phi-\Omega^h}{g} \text{ for } \hat{q} \geq \hat{q}_{l,\min} \tag{A-3}$$

$$K_1 \equiv \frac{\Psi+\nu}{g},\ K_2 \equiv \frac{\Pi}{g},\ K_3 \equiv 0 \text{ and } K_4 \equiv \frac{\tilde{w}^s\phi-\Omega^h}{g} \text{ for } \hat{q}_{l,\min} > \hat{q} \geq \hat{q}_{h,\min}. \tag{A-4}$$

Then we can express the general solution for the high-type value function as

$$\begin{aligned} Y^h(\hat{q}) &= \hat{q}^{-K_1}\left(\int\left[K_2\hat{q}^{K_1+\varepsilon-2}+K_3\hat{q}^{K_1-\frac{\Psi+g}{g}}-K_4\hat{q}^{K_1-1}\right]d\hat{q}+D\right) \\ &= \frac{K_2\hat{q}^{\varepsilon-1}}{K_1+\varepsilon-1}+\frac{K_3\hat{q}^{1-\frac{\Psi+g}{g}}}{K_1+1-\frac{\Psi+g}{g}}-\frac{K_4}{K_1}+D\hat{q}^{-K_1}. \end{aligned} \tag{A-5}$$

To find the constant of integration D, we use $Y^h(\hat{q}_{h,\min})=0$, which yields

$$D=-\frac{K_2\hat{q}_{h,\min}^{K_1+\varepsilon-1}}{K_1+\varepsilon-1}-\frac{K_3\hat{q}_{h,\min}^{K_1+1-\frac{\Psi+g}{g}}}{K_1+1-\frac{\Psi+g}{g}}+\frac{K_4\hat{q}_{h,\min}^{K_1}}{K_1} \text{ for } \hat{q}\in\left[\hat{q}_{h,\min},\hat{q}_{l,\min}\right].$$

Then we can express the value function as

$$Y^h(\hat{q})=\left\{\begin{array}{c} \frac{K_2\hat{q}^{\varepsilon-1}}{K_1+\varepsilon-1}+\frac{K_3\hat{q}^{1-\frac{\Psi+g}{g}}}{K_1+1-\frac{\Psi+g}{g}}-\frac{K_4}{K_1} \\ +\left[-\frac{K_2\hat{q}_{h,\min}^{\varepsilon-1}}{K_1+\varepsilon-1}-\frac{K_3\hat{q}_{h,\min}^{1-\frac{\Psi+g}{g}}}{K_1+1-\frac{\Psi+g}{g}}+\frac{K_4}{K_1}\right]\left(\frac{\hat{q}_{h,\min}}{\hat{q}}\right)^{K_1} \end{array}\right\}=\begin{array}{c} \frac{K_2\hat{q}^{\varepsilon-1}}{K_1+\varepsilon-1}\left[1-\left(\frac{\hat{q}_{h,\min}}{\hat{q}}\right)^{K_1+\varepsilon-1}\right] \\ +\frac{K_3\hat{q}^{-\frac{\Psi}{g}}}{K_1-\frac{\Psi}{g}}\left[1-\left(\frac{\hat{q}_{h,\min}}{\hat{q}}\right)^{K_1-\frac{\Psi}{g}}\right] \\ -\frac{K_4}{K_1}\left[1-\left(\frac{\hat{q}_{h,\min}}{\hat{q}}\right)^{K_1}\right] \end{array}.$$

Then from (A-4), we have that for $\hat{q}\in[\hat{q}_{h,\min},\hat{q}_{l,\min}]$,

$$Y^h(\hat{q})=\frac{\Pi\hat{q}^{\varepsilon-1}}{\Psi+\nu+(\varepsilon-1)g}\left(1-\left(\frac{\hat{q}_{h,\min}}{\hat{q}}\right)^{\frac{\Psi+\nu+(\varepsilon-1)g}{g}}\right)+\frac{\Omega^h-\tilde{w}^s\phi}{\Psi+\nu}\left(1-\left(\frac{\hat{q}_{h,\min}}{\hat{q}}\right)^{\frac{\Psi+\nu}{g}}\right).$$

Intuitively, because product lines with relative quality $\hat{q}\in[\hat{q}_{h,\min},\hat{q}_{l,\min}]$ immediately become obsolete when operated by low-type firms, but not by high-type firms, the flow rate of transitioning from high-type to low-type, ν, becomes part of the effective discount rate in this range.

For $\hat{q}\geq\hat{q}_{l,\min}$, the appropriate values for K's from (A-3) delivers (A-5) as

$$Y^h(\hat{q})=\frac{\Pi\hat{q}^{\varepsilon-1}}{\Psi+(\varepsilon-1)g}\left(1-\left(\frac{\hat{q}_{l,\min}}{\hat{q}}\right)^{\frac{\Psi+(\varepsilon-1)g}{g}}\right)+\frac{\Omega^l-\tilde{w}^s\phi}{\Psi}\left(1-\left(\frac{\hat{q}_{l,\min}}{\hat{q}}\right)^{\frac{\Psi}{g}}\right)+\frac{\Omega^h-\Omega^l}{\Psi+\nu}+D\hat{q}^{-\frac{\Psi+\nu}{g}}$$

We also have the boundary condition

$$\Upsilon^h\left(\hat{q}_{l,\min}\right)=\frac{\Pi\hat{q}_{l,\min}^{\varepsilon-1}}{\Psi+\nu+(\varepsilon-1)g}\left(1-\left(\frac{\hat{q}_{h,\min}}{\hat{q}_{l,\min}}\right)^{\frac{\Psi+\nu+(\varepsilon-1)g}{g}}\right)+\frac{\Omega^h-\tilde{w}^s\phi}{\Psi+\nu}\left(1-\left(\frac{\hat{q}_{h,\min}}{\hat{q}_{l,\min}}\right)^{\frac{\Psi+\nu}{g}}\right). \tag{A-6}$$

Hence, the constant of integration for $\hat{q}\geq\hat{q}_{l,\min}$ must satisfy (A-6). Next using (A-3) and (A-5), $\Upsilon^h\left(\hat{q}_{l,\min}\right)$ for $\hat{q}\geq\hat{q}_{l,\min}$ can be computed as

$$\begin{aligned}\Upsilon^h\left(\hat{q}_{l,\min}\right) &= \frac{K_2\hat{q}_{l,\min}^{\varepsilon-1}}{K_1+\varepsilon-1}+\frac{K_3\hat{q}_{l,\min}^{1-\frac{\Psi+g}{g}}}{K_1+1-\frac{\Psi+g}{g}}-\frac{K_4}{K_1}+D\hat{q}_{l,\min}^{-K_1}\\ &= \frac{\left(\Pi+\nu\xi_4\right)\hat{q}_{l,\min}^{\varepsilon-1}}{\Psi+\nu+g\left(\varepsilon-1\right)}+\xi_5\hat{q}_{l,\min}^{-\frac{\Psi}{g}}-\frac{\nu\xi_6+\tilde{w}^s\phi-\Omega^h}{\Psi+\nu}+D\hat{q}_{l,\min}^{-\frac{\Psi+\nu}{g}},\end{aligned} \tag{A-7}$$

which must be equal to (A-6). Equating (A-6) to (A-7), we get

$$D=\left\{\begin{array}{c}-\frac{\Pi}{\Psi+\nu+(\varepsilon-1)g}\hat{q}_{h,\min}^{\frac{\Psi+\nu+(\varepsilon-1)g}{g}}+\frac{\tilde{w}^s\phi-\Omega^h}{\Psi+\nu}\hat{q}_{h,\min}^{\frac{\Psi+\nu}{g}}\\ -\frac{\nu\xi_4}{\Psi+\nu+g(\varepsilon-1)}\hat{q}_{l,\min}^{\frac{\Psi+\nu+(\varepsilon-1)g}{g}}-\xi_5\hat{q}_{l,\min}^{\frac{\nu}{g}}+\frac{\nu\xi_6}{\Psi+\nu}\hat{q}_{l,\min}^{\frac{\Psi+\nu}{g}}\end{array}\right\}.$$

Hence

$$\hat{q}^{-\frac{\Psi+\nu}{g}}D=\left\{\begin{array}{c}\frac{\Pi\hat{q}^{\varepsilon-1}}{\Psi+\nu+g(\varepsilon-1)}\left(1-\left(\frac{\hat{q}_{h,\min}}{\hat{q}}\right)^{\frac{\Psi+\nu+(\varepsilon-1)g}{g}}\right)-\frac{\Pi\hat{q}^{\varepsilon-1}}{\Psi+\nu+g(\varepsilon-1)}\left(1-\left(\frac{\hat{q}_{l,\min}}{\hat{q}}\right)^{\frac{\Psi+\nu+(\varepsilon-1)g}{g}}\right)\\ +\frac{\Omega^l-\Omega^h}{\Psi+\nu}\\ -\frac{\tilde{w}^s\phi-\Omega^h}{\Psi+\nu}\left(1-\left(\frac{\hat{q}_{h,\min}}{\hat{q}}\right)^{\frac{\Psi+\nu}{g}}\right)+\frac{\tilde{w}^s\phi-\Omega^l}{\Psi+\nu}\left(1-\left(\frac{\hat{q}_{l,\min}}{\hat{q}}\right)^{\frac{\Psi+\nu}{g}}\right)\end{array}\right\}.$$

Therefore, for $\hat{q}\geq\hat{q}_{l,\min}$ we have

$$\Upsilon^h\left(\hat{q}\right)=\left\{\begin{array}{c}\frac{\Pi\hat{q}^{\varepsilon-1}}{\Psi+\nu+g(\varepsilon-1)}\left(1-\left(\frac{\hat{q}_{h,\min}}{\hat{q}}\right)^{\frac{\Psi+\nu+(\varepsilon-1)g}{g}}\right)+\frac{\Omega^h-\tilde{w}^s\phi}{\Psi+\nu}\left(1-\left(\frac{\hat{q}_{h,\min}}{\hat{q}}\right)^{\frac{\Psi+\nu}{g}}\right)\\ \frac{\Pi\hat{q}^{\varepsilon-1}}{\Psi+(\varepsilon-1)g}\left(1-\left(\frac{\hat{q}_{l,\min}}{\hat{q}}\right)^{\frac{\Psi+(\varepsilon-1)g}{g}}\right)+\frac{\Omega^l-\tilde{w}^s\phi}{\Psi}\left(1-\left(\frac{\hat{q}_{l,\min}}{\hat{q}}\right)^{\frac{\Psi}{g}}\right)\\ -\left(\frac{\Pi\hat{q}^{\varepsilon-1}}{\Psi+\nu+g(\varepsilon-1)}\left(1-\left(\frac{\hat{q}_{l,\min}}{\hat{q}}\right)^{\frac{\Psi+\nu+(\varepsilon-1)g}{g}}\right)+\frac{\Omega^l-\tilde{w}^s\phi}{\Psi+\nu}\left(1-\left(\frac{\hat{q}_{l,\min}}{\hat{q}}\right)^{\frac{\Psi+\nu}{g}}\right)\right)\end{array}\right\}.$$

Finally, we need to determine the values for the exit thresholds $\hat{q}_{l,\min}$ and $\hat{q}_{h,\min}$. Using the above differential equations we get

$$\left.\frac{\partial\Upsilon^l\left(\hat{q}\right)}{\partial t}\right|_{\hat{q}=\hat{q}_{l,\min}}=\frac{1}{g}\left(\Pi\hat{q}_{l,\min}^{\varepsilon-2}+\frac{\Omega^l-\tilde{w}^s\phi}{\hat{q}_{l,\min}}\right).$$

From the smooth-pasting condition we get

$$\left.\frac{\partial \Upsilon^{l}\left(\hat{q}\right)}{\partial \hat{q}}\right|_{\hat{q}=\hat{q}_{l,\min}}=0 \implies \hat{q}_{l,\min}=\left(\frac{\tilde{w}^{s}\phi-\Omega^{l}}{\Pi}\right)^{\frac{1}{\varepsilon-1}}.$$

Similarly, we also have

$$\left.\frac{\partial \Upsilon^{h}\left(\hat{q}\right)}{\partial \hat{q}}\right|_{\hat{q}=\hat{q}_{h,\min}}=\frac{\Pi}{\Psi+\nu+\left(\varepsilon-1\right)g}\left(\left(\varepsilon-1\right)\hat{q}^{\varepsilon-2}+\frac{\Psi+\nu}{g}\hat{q}_{h,\min}^{\frac{\Psi+\nu+(\varepsilon-1)g}{g}}\hat{q}^{-\frac{\Psi+\nu}{g}-1}\right)-\frac{\tilde{w}^{s}\phi-\Omega^{h}}{g}\hat{q}_{h,\min}^{\frac{\Psi+\nu}{g}}\hat{q}^{-\frac{\Psi+\nu}{g}-}$$

and $\left.\frac{\partial \Upsilon^{h}(\hat{q})}{\partial \hat{q}}\right|_{\hat{q}=\hat{q}_{h,\min}}=0$ implies

$$\hat{q}_{h,\min}=\left(\frac{\tilde{w}^{s}\phi-\Omega^{h}}{\Pi}\right)^{\frac{1}{\varepsilon-1}}.$$

■

Lemma 3 *Let F denote the overall relative productivity distribution, including both active and inactive product lines. In stationary equilibrium, it satisfies the following differential equation:*

$$g\hat{q}f\left(\hat{q}\right)=\tau\left[F\left(\hat{q}\right)-F(\hat{q}-\lambda\bar{\hat{q}})\right],$$

where $\tau=\Phi^{h}x^{h}+\Phi^{l}x^{l}+x^{entry}$ and $\bar{\hat{q}}=\int_{0}^{\infty}\hat{q}f\left(\hat{q}\right)d\hat{q}$. Moreover let $\tilde{F}_k$ denote the (unnormalized) distribution of relative productivities of active product lines, owned by type $k\in\{h,l\}$. In stationary equilibrium, they satisfy

$$g\hat{q}\tilde{f}_{h}(\hat{q})=g\hat{q}_{h,\min}\tilde{f}_{h}(\hat{q}_{h,\min})+\left(\tau^{l}+\varphi+\nu\right)\tilde{F}_{h}(\hat{q})-\tau^{h}\left[F(\hat{q}-\lambda\bar{\hat{q}})-F(\hat{q}_{h,min}-\lambda\bar{\hat{q}})-\tilde{F}_{h}(\hat{q})\right]$$
$$g\hat{q}\tilde{f}_{l}(\hat{q})=g\hat{q}_{l,\min}\tilde{f}_{l}(\hat{q}_{l,\min})+\left(\tau^{h}+\varphi\right)\tilde{F}_{l}(\hat{q})-\tau^{l}\left[F(\hat{q}-\lambda\bar{\hat{q}})-F(\hat{q}_{l,min}-\lambda\bar{\hat{q}})-\tilde{F}_{l}(\hat{q})\right]-\nu\left[\tilde{F}_{h}(\hat{q})-\tilde{F}_{h}(\hat{q}_{l}\right.$$

where $\tau^{l}=\Phi^{l}x^{l}+\left(1-\alpha\right)x^{entry}$ and $\tau^{h}=\Phi^{h}x^{h}+\alpha x^{entry}$. The measure of active product lines are given by

$$\Phi^{k}=\tilde{F}_{k}(\infty),\ k\in\{h,l\}.$$

Proof of Lemma 3. In a stationary equilibrium inflows and outflows into different parts of the distributions have to be equal. First consider overall productivity distribution F. Given a time interval of Δt, this implies that $F_t\left(\hat{q}\right)=F_{t+\Delta t}(\hat{q})$,

$$F_{t}\left(\hat{q}\right)=F_{t}\left(\hat{q}(1+g\Delta t)\right)-\tau\Delta t\left[F_{t}\left(\hat{q}\right)-F_{t}(\hat{q}-\lambda\bar{\hat{q}})\right]$$

Next, subtract $F_t\left(\hat{q}\left(1+g\Delta t\right)\right)$ from both sides, multiply both sides by -1, divide again sides by Δt, and take the limit as $\Delta t\to 0$, so that

$$\lim_{\Delta t \to 0} \frac{F\left(\hat{q}\left(1+g\Delta t\right)\right) - F\left(\hat{q}\right)}{\Delta t} = g\hat{q}f\left(\hat{q}\right).$$

Using this last expression delivers

$$g\hat{q}f(\hat{q}) = \tau\left[F\left(\hat{q}\right) - F(\hat{q} - \lambda\bar{\hat{q}})\right].$$

Similarly, for active product line distributions $\tilde{F}_k$, we can write

$$\tilde{F}_{h,t}(\hat{q}) = \tilde{F}_{h,t}(\hat{q}(1+g\Delta t)) - \tilde{F}_{h,t}(\hat{q}_{h,\min}(1+g\Delta t)) + \tau^h \Delta t\left[F_t(\hat{q}-\lambda\bar{\hat{q}}) - \tilde{F}_{h,t}(\hat{q}) - F_t(\hat{q}_{h,min}-\lambda\bar{\hat{q}})\right] - \left(\tau^l + \varphi + \nu\right)\Delta t \tilde{F}_{h,t}(\hat{q})$$

$$\tilde{F}_{l,t}(\hat{q}) = \tilde{F}_{l,t}(\hat{q}(1+g\Delta t)) - \tilde{F}_{l,t}(\hat{q}_{l,\min}(1+g\Delta t)) + \tau^l \Delta t\left[F_t(\hat{q}-\lambda\bar{\hat{q}}) - \tilde{F}_{l,t}(\hat{q}) - F_t(\hat{q}_{l,min}-\lambda\bar{\hat{q}})\right] - \left(\tau^h + \varphi\right)\Delta t \tilde{F}_{l,t}(\hat{q}) + \nu\Delta t\left[\tilde{F}_{h,t}(\hat{q}) - \tilde{F}_{h,t}(\hat{q}_{l,min})\right].$$

Again, by subtracting $\tilde{F}_{k,t}\left(\hat{q}\left(1+g\Delta t\right)\right) - \tilde{F}_{k,t}(\hat{q}_{k,\min}(1+g\Delta t))$ from both sides, dividing by -Δt, and taking the limit as $\Delta t \to 0$, we get the desired equations for $k \in \{h, l\}$ in Lemma 3. ■

Proof of Proposition 2. As shown in Lemma 3, overall productivity distribution satisfies

$$\hat{q}f(\hat{q}) = \frac{\tau}{g}\left[F(\hat{q}) - F\left(\hat{q} - \lambda\bar{\hat{q}}\right)\right]$$

By integrating both sides over the domain, we get

$$\mathbb{E}(\hat{q}) \equiv \int_0^\infty \hat{q}f(\hat{q})d\hat{q} = \frac{\tau}{g}\int_0^\infty \left[F(\hat{q}) - F\left(\hat{q} - \lambda\bar{\hat{q}}\right)\right]d\hat{q}$$

We can write above equation as follows

$$\mathbb{E}(\hat{q}) = \frac{\frac{\tau}{g}}{1+\frac{\tau}{g}}\int_0^\infty \left[1 - F\left(\hat{q} - \lambda\bar{\hat{q}}\right)\right]d\hat{q}.$$

as $\int_0^\infty \left[1 - F(\hat{q})\right]d\hat{q} = \mathbb{E}(\hat{q})$.

By changing of variable as $x = \hat{q} - \lambda\bar{\hat{q}}$, which implies $dx = d\hat{q}$, we have

$$\mathbb{E}(\hat{q}) = \frac{\frac{\tau}{g}}{1+\frac{\tau}{g}}\int_{-\lambda\bar{\hat{q}}}^\infty \left[1 - F\left(x\right)\right]dx = \frac{\tau}{g}\lambda\bar{\hat{q}}$$

Last equality follows from the fact that $F(x) = 0$ for $x \leq 0$. In equilibrium we have, $\bar{\hat{q}} = \mathbb{E}(\hat{q})$. Therefore

$$g = \tau\lambda.$$

■

Appendix B: Estimation Results from the Robustness Exercises

B-1 Employment Weighted Sample

Table B-1: Estimated Parameters

#	Parameter	Description	Value
1.	ϕ	Fixed cost of operation	0.168
2.	θ^H	Innovative capacity of high-type firms	2.209
3.	θ^L	Innovative capacity of low-type firms	1.532
4.	θ^E	Innovative capacity of entrants	0.025
5.	α	Probability of being high-type entrant	0.917
6.	ν	Transition rate from high-type to low-type	0.258
7.	λ	Innovation step size	0.101
8.	φ	Exogenous destruction rate	0.039

Table B-2: Model and Data Moments

#	Moments	Model	Data	#	Moments	Model	Data
1.	Firm exit (small-young)	0.104	0.107	10.	Sales growth (small-young)	0.079	0.079
2.	Firm exit (small-old)	0.102	0.077	11.	Sales growth (small-old)	0.020	0.019
3.	Firm exit (large-old)	0.039	0.036	12.	Sales growth (large-old)	-0.023	-0.022
4.	Trans. from large to small	0.025	0.010	13.	R&D to sales (small-young)	0.100	0.075
5.	Trans. from small to large	0.036	0.014	14.	R&D to sales (small-old)	0.066	0.048
6.	Prob. of small (cond on entry)	0.795	0.753	15.	R&D to sales (large-old)	0.066	0.055
7.	Emp. growth (small-young)	0.078	0.073	16.	5-year Entrant Share	0.361	0.393
8.	Emp. growth (small-old)	0.020	0.028	17.	Fixed cost-R&D labor ratio	3.284	5.035
9.	Emp. growth (large-old)	-0.023	-0.033	18.	Aggregate growth	0.022	0.022

B-2 Organic Sample that Excludes M&A Activities

Table B-3: Parameter Estimates

#	Parameter	Description	Value
1.	ϕ	Fixed cost of operation	0.233
2.	θ^H	Innovative capacity of high-type firms	1.708
3.	θ^L	Innovative capacity of low-type firms	1.480
4.	θ^E	Innovative capacity of entrants	0.023
5.	α	Probability of being high-type entrant	0.806
6.	ν	Transition rate from high-type to low-type	0.213
7.	λ	Innovation step size	0.137
8.	φ	Exogenous destruction rate	0.030

Table B-4: Model and Data Moments

#	Moments	Model	Data	#	Moments	Model	Data
1.	Firm exit (small-young)	0.097	0.112	10.	Sales growth (small-young)	0.086	0.102
2.	Firm exit (small-old)	0.091	0.067	11.	Sales growth (small-old)	0.049	0.014
3.	Firm exit (large-old)	0.030	0.022	12.	Sales growth (large-old)	-0.003	-0.003
4.	Trans. from large to small	0.021	0.009	13.	R&D to sales (small-young)	0.070	0.048
5.	Trans. from small to large	0.037	0.010	14.	R&D to sales (small-old)	0.065	0.061
6.	Prob. of small (cond on entry)	0.873	0.899	15.	R&D to sales (large-old)	0.057	0.035
7.	Emp. growth (small-young)	0.088	0.106	16.	5-year Entrant Share	0.319	0.381
8.	Emp. growth (small-old)	0.049	0.028	17.	Fixed cost-R&D labor ratio	4.592	5.035
9.	Emp. growth (large-old)	-0.002	-0.002	18.	Aggregate growth	0.022	0.022

B-3 Baseline Estimation without R&D Moments

Table B-5: Estimated Parameters

#	Parameter	Description	Value
1.	ϕ	Fixed cost of operation	0.215
2.	θ^H	Innovative capacity of high-type firms	1.711
3.	θ^L	Innovative capacity of low-type firms	1.407
4.	θ^E	Innovative capacity of entrants	0.030
5.	α	Probability of being high-type entrant	0.894
6.	ν	Transition rate from high-type to low-type	0.207
7.	λ	Innovation step size	0.130
8.	φ	Exogenous destruction rate	0.035

Table B-6: Model and Data Moments

#	Moments	Model	Data	#	Moments	Model	Data
1.	Firm exit (small-young)	0.098	0.107	10.	Sales growth (small-young)	0.095	0.107
2.	Firm exit (small-old)	0.092	0.077	11.	Sales growth (small-old)	0.046	0.024
3.	Firm exit (large-old)	0.035	0.036	12.	Sales growth (large-old)	-0.005	-0.003
4.	Trans. from large to small	0.021	0.010	13.	R&D to sales (small-young)	-	-
5.	Trans. from small to large	0.037	0.014	14.	R&D to sales (small-old)	-	-
6.	Prob. of small (cond on entry)	0.853	0.753	15.	R&D to sales (large-old)	-	-
7.	Emp. growth (small-young)	0.096	0.106	16.	5-year Entrant Share	0.333	0.393
8.	Emp. growth (small-old)	0.046	0.035	17.	Fixed cost-R&D labor ratio	4.263	5.035
9.	Emp. growth (large-old)	-0.005	-0.005	18.	Aggregate growth	0.022	0.022

Table B-7: Excluding R&D Moments

x^{entry}	x^l	x^h	Φ^l	Φ^h	$\hat{q}_{l,\min}$	$\hat{q}_{h,\min}$	$\frac{L^{R\&D}}{L^S}$	τ	g	Wel
Panel A. Baseline										
0.62	26.04	35.71	56.67	5.15	146.54	133.84	19.62	17.24	2.23	100.00
Panel B. Social Planner										
0.75	26.52	44.29	10.46	41.24	217.08	29.67	32.88	21.79	2.82	103.63
Panel C. Incumbent R&D and Operation ($s_i = -2\%, s_o = -69\%$)										
0.76	31.10	43.34	49.08	7.52	159.59	147.21	26.57	19.29	2.50	101.38

B-4 Manufacturing Sample

Table B-8: Parameter Estimates

#	Parameter	Description	Value
1.	ϕ	Fixed cost of operation	0.448
2.	θ^H	Innovative capacity of high-type firms	0.277
3.	θ^L	Innovative capacity of low-type firms	0.058
4.	θ^E	Innovative capacity of entrants	0.017
5.	α	Probability of being high-type entrant	0.699
6.	ν	Transition rate from high-type to low-type	0.460
7.	λ	Innovation step size	0.452
8.	φ	Exogenous destruction rate	0.044

Table B-9: Model and Data Moments

#	Moments	Model	Data	#	Moments	Model	Data
1.	Firm exit (small-young)	0.096	0.081	10.	Sales growth (small-young)	0.018	0.018
2.	Firm exit (small-old)	0.103	0.059	11.	Sales growth (small-old)	0.001	0.001
3.	Firm exit (large-old)	0.053	0.037	12.	Sales growth (large-old)	-0.059	0.008
4.	Trans. from large to small	0.037	0.011	13.	R&D to sales (small-young)	-	-
5.	Trans. from small to large	0.020	0.009	14.	R&D to sales (small-old)	-	-
6.	Prob. of small (cond on entry)	0.530	0.669	15.	R&D to sales (large-old)	-	-
7.	Emp. growth (small-young)	0.018	0.020	16.	5-year Entrant Share	0.390	0.425
8.	Emp. growth (small-old)	0.008	-0.003	17.	Fixed cost-R&D labor ratio	4.955	5.035
9.	Emp. growth (large-old)	-0.055	-0.008	18.	Aggregate growth	0.019	0.019

B-5 Model with Unskilled Overhead Labor

Table B-10: Parameter Estimates

#	Parameter	Description	Value
1.	ϕ	Fixed cost of operation	0.219
2.	θ^H	Innovative capacity of high-type firms	1.925
3.	θ^L	Innovative capacity of low-type firms	1.404
4.	θ^E	Innovative capacity of entrants	0.030
5.	α	Probability of being high-type entrant	0.883
6.	ν	Transition rate from high-type to low-type	0.196
7.	λ	Innovation step size	0.140
8.	φ	Exogenous destruction rate	0.049
9.	β	Fraction of managers with a college degree or above	0.457

Table B-11: Model and Data Moments

#	Moments	Model	Data	#	Moments	Model	Data
1.	Firm exit (small-young)	0.099	0.107	10.	Sales growth (small-young)	0.107	0.107
2.	Firm exit (small-old)	0.098	0.077	11.	Sales growth (small-old)	0.048	0.024
3.	Firm exit (large-old)	0.046	0.036	12.	Sales growth (large-old)	-0.006	-0.003
4.	Trans. from large to small	0.020	0.010	13.	R&D to sales (small-young)	0.108	0.064
5.	Trans. from small to large	0.039	0.014	14.	R&D to sales (small-old)	0.076	0.059
6.	Prob. of small (cond on entry)	0.807	0.753	15.	R&D to sales (large-old)	0.065	0.037
7.	Emp. growth (small-young)	0.104	0.106	16.	5-year Entrant Share	0.369	0.393
8.	Emp. growth (small-old)	0.047	0.035	17.	Fixed cost-R&D labor ratio	5.656	5.035
9.	Emp. growth (large-old)	-0.005	-0.005	18.	Aggregate growth	0.022	0.022

B-6 Model with Reallocation Cost

Table B-12: Parameter Estimates

#	Parameter	Description	Value
1.	ϕ	Fixed cost of operation	0.201
2.	θ^H	Innovative capacity of high-type firms	1.840
3.	θ^L	Innovative capacity of low-type firms	1.287
4.	θ^E	Innovative capacity of entrants	0.017
5.	α	Probability of being high-type entrant	0.960
6.	ν	Transition rate from high-type to low-type	0.300
7.	λ	Innovation step size	0.134
8.	φ	Exogenous destruction rate	0.038

Table B-13: Model and Data Moments

#	Moments	Model	Data	#	Moments	Model	Data
1.	Firm exit (small-young)	0.093	0.107	10.	Sales growth (small-young)	0.103	0.107
2.	Firm exit (small-old)	0.088	0.077	11.	Sales growth (small-old)	0.033	0.024
3.	Firm exit (large-old)	0.036	0.036	12.	Sales growth (large-old)	-0.005	-0.003
4.	Trans. from large to small	0.020	0.010	13.	R&D to sales (small-young)	0.090	0.064
5.	Trans. from small to large	0.037	0.014	14.	R&D to sales (small-old)	0.058	0.059
6.	Prob. of small (cond on entry)	0.841	0.753	15.	R&D to sales (large-old)	0.052	0.037
7.	Emp. growth (small-young)	0.099	0.106	16.	5-year Entrant Share	0.321	0.393
8.	Emp. growth (small-old)	0.033	0.035	17.	Fixed cost-R&D labor ratio	4.237	5.035
9.	Emp. growth (large-old)	-0.005	-0.005	18.	Aggregate growth	0.022	0.022

B-7 Model with Three Types

Table B-14: Parameter Estimates

#	Parameter	Description	Value
1.	ϕ	Fixed cost of operation	0.229
2.	θ^H	Innovative capacity of high-type firms	1.802
3.	θ^M	Innovative capacity of medium-type firms	1.753
4.	θ^L	Innovative capacity of low-type firms	1.381
5.	θ^E	Innovative capacity of entrants	0.023
6.	α_H	Probability of being high-type entrant	0.105
7.	α_M	Probability of being medium-type entrant	0.855
8.	ν	Transition rate to low-type	0.215
9.	λ	Innovation step size	0.134
10.	φ	Exogenous destruction rate	0.036

Table B-15: Model and Data Moments

#	Moments	Model	Data	#	Moments	Model	Data
1.	Firm exit (small-young)	0.096	0.107	10.	Sales growth (small-young)	0.100	0.107
2.	Firm exit (small-old)	0.092	0.077	11.	Sales growth (small-old)	0.037	0.024
3.	Firm exit (large-old)	0.035	0.036	12.	Sales growth (large-old)	-0.005	-0.003
4.	Trans. from large to small	0.021	0.010	13.	R&D to sales (small-young)	0.083	0.064
5.	Trans. from small to large	0.037	0.014	14.	R&D to sales (small-old)	0.063	0.059
6.	Prob. of small (cond on entry)	0.849	0.753	15.	R&D to sales (large-old)	0.056	0.037
7.	Emp. growth (small-young)	0.099	0.106	16.	5-year Entrant Share	0.329	0.393
8.	Emp. growth (small-old)	0.038	0.035	17.	Fixed cost-R&D labor ratio	4.386	5.035
9.	Emp. growth (large-old)	-0.005	-0.005	18.	Aggregate growth	0.022	0.022