Incorporation, Selection and Firm Dynamics: A Quantitative Exploration *

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Abstract

This paper studies how incorporation, which provides limited liability to firm owners, affects firm dynamics and macroeconomy. I propose an endogenous growth model of firm dynamics with endogenous entry and exit, where firms spend resources to improve their productivity and choose whether to incorporate or not. Incorporation provides liability protection which ensures that firm value is bounded from below, at the expense of high set-up and maintaining cost. An important model feature is that firms have heterogeneous (high and low) types which differ in their capacity to improve productivity. This heterogeneity allows for the possibility of selection as high-type firms, who have higher growth potential, benefit more from incorporation. I calibrate the model by using Danish firm-level data, specifically exploiting the heterogeneity in exit rates by age conditional on size to identify firm types in growth potential and therefore selection. The quantitative results suggest that both treatment and selection effects of incorporation are important and accounting for firm heterogeneity is quantitatively relevant in explaining the observed better performance of incorporated firms. Conditional on the firm type, incorporated firms choose an expansion rate, the rate at which firms improve their productivity, 50% higher than unincorporated firms do on average. Upon entry, 90% (15%) of the incorporated (unincorporated) firms are high-types, which are estimated twice as efficient as low-types in improving their productivity. This underlines a significant selection effect which is more pronounced among incumbents as the exit rate of high-type firms is lower. In a counterfactual economy where the incorporation decision is randomized within firm types, the productivity growth decreases from 3% to 2.7% and the difference in the average size of incorporated and unincorporated firms decreases by 32%. I find significant welfare gains from subsidizing incorporated firms and large welfare losses from removing incorporation choice. These welfare results are largely driven by the change in the degree of selection, i.e. the change in the composition of firm types.

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1. Introduction

An extensive empirical literature has documented striking differences across firms. While many firms fail in their early years of existence and most of those that survive do not grow, others grow rapidly and significantly contribute to job creation and aggregate productivity growth. This reflects substantial firm heterogeneity in many aspects, one of which is the legal form they choose to operate in. For example in the U.S. roughly half of all business owners prefer to shield themselves against the downside risks by attaining limited liability through incorporating their businesses. How does limited liability affect firm behavior and the macroeconomy? How does the choice of legal form interact with ex-ante and ex-post firm heterogeneity?

To answer these questions, this paper proposes a macroeconomic model of firm dynamics with endogenous entry and exit, where entrepreneurs choose whether or not to incorporate their firms. In the model, firms invest in resources to improve their productivities which determine their profitability and contribute to economic growth. Firms have heterogeneous (high and low) types that differ in their efficiency to improve productivity. In other words, firms are heterogeneous in terms of their growth potential. Successful entrepreneurs increase their firm productivity and stay in the economy, whereas unsuccessful ones end up exiting the economy, either endogenously due to the deterioration in their profitability or due to exogenous shocks that render the firm unproductive. Firms are subject to an exit cost which is proportional to their size. Due to this cost, the firm value falls below zero in the case of exit. By paying a sunk cost, entrepreneurs can incorporate their firms to ensure that their losses are limited to the initial cost of setting up the firm. In other words, incorporation provides insurance to the owner by bounding the firm value from below.

This environment underlines two main effects that generate differences in firm dynamics between incorporated and unincorporated firms. The first one is a treatment effect of incorporation: since incorporation protects firms from downside risks, it incentivizes them to invest more in improving their productivity, subsequently grow large and exit less often. The second one is a selection effect due to the presence

of firm heterogeneity: entrepreneurs with higher growth potential (i.e. more efficient in improving productivity) are more likely to choose incorporation as it is more valuable to large firms. The strength of this selection effect is determined by the interplay between endogenous entry, investment, and exit decisions.

To quantify the importance of these effects and study their macroeconomic implications, I calibrate the model to firm-level micro data from Denmark. Calibration targets several key empirical moments of firm dynamics for incorporated and unincorporated firms. Specifically, the calibrated model is able to quantitatively match the observed differences between incorporated and unincorporated firms: incorporated firms have higher employment upon entry, grow faster, and exit less often conditional on their size and age, compared to unincorporated firms. Furthermore, to validate the model, I show that a variety of moments that are not targeted in the estimation are in line with the data.

My calibration strategy exploits the heterogeneity in firms exit rates by age conditional on size and legal form to identify firm heterogeneity in growth potential. The model implies that without this firm type heterogeneity, the likelihood of exit would be independent of age conditional on size. In data, however, such conditional exit rates are decreasing in firm age for both incorporated and unincorporated firms. My framework rationalizes this pattern through the interaction between firm heterogeneity and endogenous selection in that the share of firms with high growth potential, which have lower exit rates conditional on size, increases within a given cohort as the cohort ages.

The quantitative results suggest that both treatment and selection effects are important and accounting for firm heterogeneity is quantitatively relevant in explaining the observed better performance of incorporated firms. Conditional on the firm type, incorporated firms choose an expansion rate, the rate at which firms improve their productivity, 50% higher than unincorporated firms do on average. This indicates a significant positive treatment effect of incorporation on firm-level productivity growth. Among entrants, 90% (15%) of the firms that choose (not) to incorporate

are high-types, highlighting a significant selection effect upon entry. Among incumbents, the selection effect becomes more pronounced where the share of high-types rises to 99% within incorporated firms. To further explore the importance of selection effect, I consider a counterfactual economy where the incorporation decision is randomized within firm types, while keeping the distribution of firm types upon entry constant. In this counterfactual economy, the difference in the average size of incorporated and unincorporated firms decreases by 32% compared to the baseline economy and the aggregate productivity growth decreases from 3% to 2.7%. Aggregate productivity growth declines mainly because the randomization of legal form decisions deteriorates the equilibrium composition of firm types.

Finally, I use the model to conduct two experiments to assess the value of incorporation. First, I consider a case where the option of incorporation is not available to the firms. The absence of incorporation not only eliminates the positive treatment effect on firms expansion rates but also mitigates the selection of high-growth potential firms in the economy. Consequently, the growth rate decreases to 2.49% and welfare decreases by 4.6% (in consumption equivalent terms). On the other hand, subsidizing the incorporated firms provides significant welfare gains. This last result is largely driven by the change in the degree of selection, i.e. the change in the composition of firm types.

Related Literature This paper is linked to a number of different literatures. Recently, the macroeconomic implications of firms legal forms have attracted some attention attention. For example, Dyrda et al. (2019) and Barro and Wheaton (2019) have investigated the recent trend of pass-through entities and C-corporations among the U.S. businesses. Specifically, Dyrda et al. (2019) focus on the trade-off entrepreneurs face between running the C-corporation versus pass-through entity in manufacturing and services sector, while Barro and Wheaton (2019) assess the effects of business taxation on choices of legal form and subsequently productivity in an empirical framework. Unlike their work, my paper focuses on the presence of limited liability

and how it affects firm dynamics and the macroeconomy.

To the best of my knowledge, the papers that come closest to mine are Herranz et al. (2015) and Short and Glover (2011). Herranz et al. (2015) point out that less risk averse entrepreneurs, because they operate larger more risky projects and therefore would gain the most from limited liability, are those who would most likely incorporate if given the option. Short and Glover (2011) propose a model where they study the bankruptcy and incorporation decisions of entrepreneurs in order to understand the types of risks faced by entrepreneurs. My paper is different from theirs in several aspects. First, their papers consider the choice of incorporation as an individual decision in the tradition of Quadrini (2000) and Cagetti and De Nardi (2006) where their main focus is entrepreneurs. By contrast, my paper mainly focuses on the firm behavior and proposes a theory that is able to capture the stylized facts of firm dynamics and growth in the economy. Second, while they consider firm productivity as exogenous, the productivity process is taken as the center stage for firm growth in my paper where firms choose the level of investment to improve their productivity. In other words, the productivity process is endogenous in my framework. Therefore, my paper allows the legal environment to affect this investment decision and hence aggregate productivity growth, which is absent in their framework.

One distinct feature of my model is it explicitly allows for heterogeneity in firms growth potential, which is essential in capturing the observed pattern of firm dynamics in data and allowing for selection effect. There is ample empirical evidence for the importance of such heterogeneity. Schoar (2010) and Decker et al. (2014) argue that some entrepreneurs are ”transformative” and have the necessary skills to expand, while subsistence entrepreneurs may simply never grow independently of the environment they operate in. Hurst and Pugsley (2012) provide evidence that many firms in the U.S. intentionally choose to remain small. On the theoretical side, Luttmer (2011), Lentz and Mortensen (2016) and Jones and Kim (2018) argue that models without heterogeneity in growth potential are unable to explain the very rapid growth of a subset of firms. Gabaix et al. (2016) argue that theories which build

on a random growth mechanism generate transition dynamics that are too slow and allowing the presence of some high-growth types can explain the observed fast rise in income inequality. Acemoglu et al. (2018) emphasize the importance of heterogeneity in innovative capacity for designing optimal R&D policies.

The rest of the paper is organized as follows: In Section 2, I describe the theoretical model. Section 3 summarizes the data that I use in the quantitative analysis and discusses the identification of the model. In Section 4, I present the calibration results, and assess the model fit based on various out-of-sample moments. In Section 5, I provide the main analysis to quantify the importance of treatment and selection effects on firm dynamics and the aggregate economy. Section 6 concludes. All proofs and additional details are contained in the Appendix.

2. Model

2.1 Preferences, Technology, and Static Allocations

The economy is in continuous time and admits a representative household with per-period log utility function

$$U_0 = \int e^{-\rho t} \ln C(t) dt \tag{1}$$

where $C(t)$ is consumption at time t and $\rho > 0$ is the discount rate. The household is populated by a continuum of individuals with measure one. Each member is endowed with one unit of labor that is supplied inelastically.

The individuals consume a unique final good $Y(t)$, which is also used for other purposes as will be discussed below. The final good is produced competitively by labor $L(t)$ and a continuum of intermediate goods over the set $\mathcal{N}(t)$, with measure Φ_t, following the production technology given below

$$Y(t) = \frac{L(t)^\beta}{1-\beta} \int_{\mathcal{N}(t)} q_j(t)^\beta y_j(t)^{1-\beta} dj \tag{2}$$

where $q_j(t)$ and $y_j(t)$ are the quality and quantity of intermediate good j, respectively. The measure of intermediate goods produced in the economy is determined endogenously through entry and exit decisions. The price of the final good is normalized to be one in every period without loss of generality. In what follows, I will drop the time subscript t whenever it does not cause any confusion.

Each intermediate good $j \in \mathcal{N}$ is produced by a single firm which monopolistically competes against other firms active in the economy. Therefore index j also refers to the firm that produces intermediate good j. These firms have access to a linear technology of the form

$$y_j = \bar{q} l_j \tag{3}$$

where l_j is the amount of labor that firm j hires for the production, and $\bar{q} \equiv \frac{\int_{\mathcal{N}} q_j dj}{\Phi}$ is the average quality in the economy. In addition to the labor cost, production requires also a fixed cost of operation $\psi \bar{q}$ at every period in terms of the final good. As will be discussed later, this fixed cost is allowed to be different for different legal forms chosen by the firm.

The maximization problem of the final goods producer generates the inverse demand $p_j = L^{\beta} q_j^{\beta} y_j^{-\beta}$. Given the production technology, each firm is faced with a constant marginal cost of production given by $w/\bar{q}$, where w is the wage rate in the economy. Therefore, for a given level of quality q_j, we can write firm j's static profit maximization problem as

$$\pi(q_j) = \max_{y_j \geq 0} \left\{ L^{\beta} q_j^{\beta} {y_j}^{1-\beta} - \frac{w}{\bar{q}} y_j \right\}.$$

where $\pi(q_j)$ is the per-period profit of firm j (before paying the fixed cost of operation) with quality q_j. The price and output level of firm follow from this maximization as

$$p_j = \frac{1}{(1-\beta)} \frac{w}{\bar{q}} \text{ and } y_j = \left[(1-\beta) \frac{\bar{q}}{w} \right]^{\frac{1}{\beta}} L q_j, \tag{4}$$

implying that the price is a constant markup over the marginal cost, and firm's op-

timal output is proportional to quality. As shown in Section 7.1 in Appendix, the maximization in the final goods sector, together with (4), implies that the wage rate is proportional to average quality in the economy. Given the production technology in (3), optimal output choice of the firm implies that labor hiring of the firm is proportional to the quality of the firm *relative to* average quality in the economy, $q/\bar{q}$. Therefore relative quality can be considered as a summary statistics for firm size.

Finally, the resulting equilibrium profits can then be written as

$$\pi(q_j) = \Pi q_j, \tag{5}$$

where $\Pi = \beta\left[(1-\beta)\right]^{\frac{1-\beta}{\beta}}\left(\frac{\bar{q}}{w}\right)^{\frac{1-\beta}{\beta}}$, i.e., profits are increasing in quality q_j. Therefore firms have profit incentives to improve their product quality, which is the source of firm growth and will be discussed in the next subsection.

2.2 Evolution of Firm Quality

Quality at the firm level evolves over time depending on the firm's investments in improving its quality. This process is modeled as a controlled stochastic process as in Akcigit and Kerr (2018) and Atkeson and Burstein (2010). In particular, I assume that by investing R in terms of final good, an incumbent firm with quality q improves its quality at the Poisson flow rate x such that

$$x = \theta\left(\frac{R}{q}\right)^{\eta} \tag{6}$$

where $\eta \in (0,1)$ and θ is the *efficiency* of the investment technology. This particular investment technology assumes that the cost required to increase the quality scales with the size of the firm. This implies that, consistent with Gibrat's law, the growth rate of sufficiently large firms (large quality) is independent of their size.

When the investment is successful, the current quality of the firm improves from

q to $q + J(q, \bar{q})$ where

$$J(q, \bar{q}) = \lambda \left[\omega \bar{q} + (1 - \omega) q\right], \quad \omega \in [0, 1]. \tag{7}$$

That is, improvement in the quality is a convex combination of current quality of the firm q and the average quality in the economy $\bar{q}$. This formulation is a generalization of Acemoglu et al. (2018), where quality improvements depend only on average quality in the economy, and Akcigit and Kerr (2018), where quality improvements are proportional to current quality of the firm.[1]

Firm Heterogeneity Firms are heterogeneous in how *efficient* they are at improving their quality. It is this heterogeneity across firms that gives rise to the possibility of selection: entrepreneurs with higher growth potential (i.e. more efficient in improving productivity) are more likely to choose incorporation as it is more valuable to large firms. Formally, I assume that firms differ in their efficiency of the investment technology θ and can be either low-type (θ_L) or high-type (θ_H). A firm's type is persistent and determined upon entry. New entrant draws its type from a Bernoulli distribution

$$\theta = \begin{cases} \theta_L & \text{with probability } \alpha \\ \theta_H & \text{with probability } 1 - \alpha \end{cases}. \tag{8}$$

where $\alpha \in (0, 1)$ and $\theta_H > \theta_L > 0$. As will be discussed later in detail, allowing this heterogeneity is not only important in quantifying the scope of firm selection into different legal forms, but is also quantitatively relevant in accounting for the firm growth and exit heterogeneity within legal forms.[2]

[1] Having average quality in (7) introduces spillovers between firms: each firm's improvement in its quality adds to the average quality, which in turn provides bigger quality improvement for all the firms in the economy. Therefore the parameter ω controls the extend of this spillover.

[2] For the relevance of this type of heterogeneity, see Acemoglu et al. (2018) in the context of optimal industrial policies and Jones and Kim (2018) in the context of top income inequality.

2.3 Entry and Exit

A unit mass of potential entrants attempts to enter the economy at any point in time. They use a similar investment technology as the incumbent firms, where the flow rate of entry x_e is related to the spending on entry efforts R_e according to $x_e = \theta_e \left(\frac{R_e}{\bar{q}}\right)^{\eta}$. Following a successful entry, the entrant first draws its initial quality from a distribution $\Psi(q)$ and its type, $\theta \in \{\theta_L, \theta_H\}$, then decides whether to incorporate its firm or not. This description implies the following optimization problem for entrants:

$$\max_{x_e} \{x_e \mathbb{E}\left(v(q,\theta)\right) - c_e(x_e, \theta_E)\} \tag{9}$$

where $\mathbb{E}\left(v(q,\theta)\right)$ is the expected value of entry (and the expectation is over the quality the successful entrants will obtain and firm type θ) and $c_e(x_e, \theta_E)$ is the cost of entry implied by the investment technology. Given that there is a unit measure of potential entrants, x_e is also equal to the total entry flow rate.

A firm's exit happens either due to (i) an exogenous death shock at Poisson rate $\kappa > 0$ or (ii) firms' choosing to exit *endogenously*: firms will voluntarily shut down when their quality is low enough such that they are no longer sufficiently profitable relative to the fixed cost of operation. When firms exit, they stop producing and their flow profits drop to zero. Importantly, I assume that firms are subject to an *exit cost* that is proportional to the quality of the firm at the time of exit, $c_E \times q$ where c_E is a parameter. This cost can be considered as a liquidation or firing cost as in Hopenhayn and Rogerson (1993) and Poschke (2009). Recall that firm's optimal output and the amount of labor are proportional to its quality as in equation (4), motivating the exit cost assumption being proportional to the quality. Importantly, the presence of exit cost drives the value of the firm to the owner below zero in the case of exit. As will be discussed in the next subsection, the presence of this exit cost creates the main motivation for a firm owner to incorporate her business.

2.4 Legal Form Choice

New entrants choose their legal form (incorporate or not) after they learn their initial quality q and their type $\theta \in \{\theta_L, \theta_H\}$. Incumbent firms have the option to switch between legal forms at arrival rate μ. Incorporation entails a sunk setup cost in terms of final good $C_I\bar{q}$.[3] In this setting, a firm with quality q and type θ chooses to incorporate if and only if

$$V_I(q;\theta) - C_I\bar{q} > V_U(q;\theta) \tag{10}$$

where V_I and V_U denote the value of an incorporated and unincorporated firm, respectively. Incorporation provides liability protection which ensures that firm owner does not suffer any losses beyond the value of the firm.[4] In other words, in the case of exit, incorporated firms do not suffer loses due to liquidation or firing costs that derive the firm value below zero. This benefit comes at the expense of set-up and maintaining cost. In short, incorporation trade-offs exit cost, which is proportional to the firm size, with higher fixed cost of operation and setup cost and it provides insurance to the firm owner by bounding the firm value from below. This trade-off and its implications on firm behavior will be more clear in the next subsection.

2.5 Firm Decision and Value Functions

I normalize all the growing variables by average quality in the economy, $\bar{q}(t)$, to keep the stationary equilibrium values constant and denote the relative quality $q/\bar{q}$ as $\hat{q}$. Moreover, let g denote the growth rate of average quality, which is also the aggregate growth rate in the economy endogenously determined in equilibrium. Then the stationary equilibrium value function for incorporated firms with relative quality $\hat{q}$ and type θ can be written as

[3] Like fixed cost of operation, setup cost of incorporation is assumed to grow with the average quality in the economy to ensure stationarity.

[4] Initial costs of starting a firm such as the cost of entry given by (9) and the setup cost of incorporation C_I are considered as sunk.

$$\rho v_I(\hat{q};\theta) = \max\left\{0, \max_{x_I \geq 0}\left[\begin{array}{c} \Pi\hat{q} - c(x_I,\hat{q},\theta) - \psi_I \\ -g\hat{q}\frac{\partial v_I(\hat{q};\theta)}{\partial \hat{q}} \\ +x_I\left[v_I\left(\hat{q}_+;\theta\right) - v_I(\hat{q};\theta)\right] \\ +\kappa\left[0 - v_I(\hat{q};\theta)\right] \\ +\mu\left[\max\{v_I(\hat{q};\theta), v_U(\hat{q};\theta)\} - v_I(\hat{q};\theta)\right] \end{array}\right]\right\} \tag{11}$$

where $v_I(\hat{q};\theta)$ is the value of the firm and $\hat{q}_+$ denotes the new level of relative quality after a successful investment.[5] Notice that the type of the firm affects the firm value through the investment cost function for quality improvements $c(x,\hat{q},\theta) = \hat{q}\left(\frac{x}{\theta}\right)^{\frac{1}{\eta}}$ implied by equation (6). This value function implicitly defines (i) a threshold level of relative quality at which firms choose to exit $\hat{q}_{min}$ and (ii) firms' optimal rate of expansion x_I which determines the rate of quality growth at the firm level.

The value function above can be interpreted as follows. Given discounting at the rate ρ, the left-hand side is the flow value of firm with relative productivity $\hat{q}$. The right-hand side includes the components that make up this flow value. The outer maximization problem determines the endogenous exit decision of the firm. Since owner of an incorporated firm is not liable losses of her business beyond the value of the firm, the value of choosing exit is zero. The first line includes the instantaneous profits, minus the cost of quality enhancing investment and the fixed costs of operation. The second line reflects the change in firm value due to the increase in the average quality in the economy, which happens at the rate g. This term accounts for the fact that as the average quality increases, the relative quality at which the firm operates declines, leading to the erosion of profits. The third line expresses the change in firm value when the firm is successful with its investment in improving quality at the rate x_I. The fourth line shows the change in value when the firm has to exit due to an exogenous death shock at the rate κ. The last line includes the change in firm value if firm decides to switch to being unincorporated.

[5] Full derivation of the value function is provided in Section 7.3 in Appendix

The value function for unincorporated firms v_U is given by

$$\rho v_U(\hat{q};\theta) = \max\left\{ -c_E\hat{q}, \max_{x_U \geq 0} \left[\begin{array}{c} \Pi\hat{q} - c(x_U,\hat{q},\theta) - \psi_U \\ -g\hat{q}\frac{\partial v_U(\hat{q};\theta)}{\partial \hat{q}} \\ +x_U\left[v_U\left(\hat{q}_+;\theta\right) - v_U(\hat{q};\theta)\right] \\ +\kappa(-c_E\hat{q} - v_U(\hat{q};\theta)) \\ +\mu\left[\max\left\{v_I(\hat{q};\theta) - C_I, v_U(\hat{q};\theta)\right\} - v_U(\hat{q};\theta)\right] \end{array} \right] \right\}. \quad (12)$$

The interpretation of the value function is same as above. The main difference between incorporated and unincorporated firms is that the value of the unincorporated firm falls below zero in the case of exit due to presence of exit cost: the owner has full liability and needs to pay the exit cost $c_E\hat{q}$. This happens either when the relative quality is too low to be profitable so that firm chooses exit endogenously or due to firm experiencing exogenous death shock at the rate κ.

The optimal expansion decision of a firm with legal form $l \in \{I,U\}$ and firm type $\theta \in \{\theta_L,\theta_H\}$ is given by

$$x_l(\hat{q};\theta) = \theta^{\frac{1}{1-\eta}}\eta^{\frac{\eta}{1-\eta}}\left[\frac{v_l\left(\hat{q}_+;\theta\right) - v_l(\hat{q};\theta)}{\hat{q}}\right]^{\frac{\eta}{1-\eta}}, \quad (13)$$

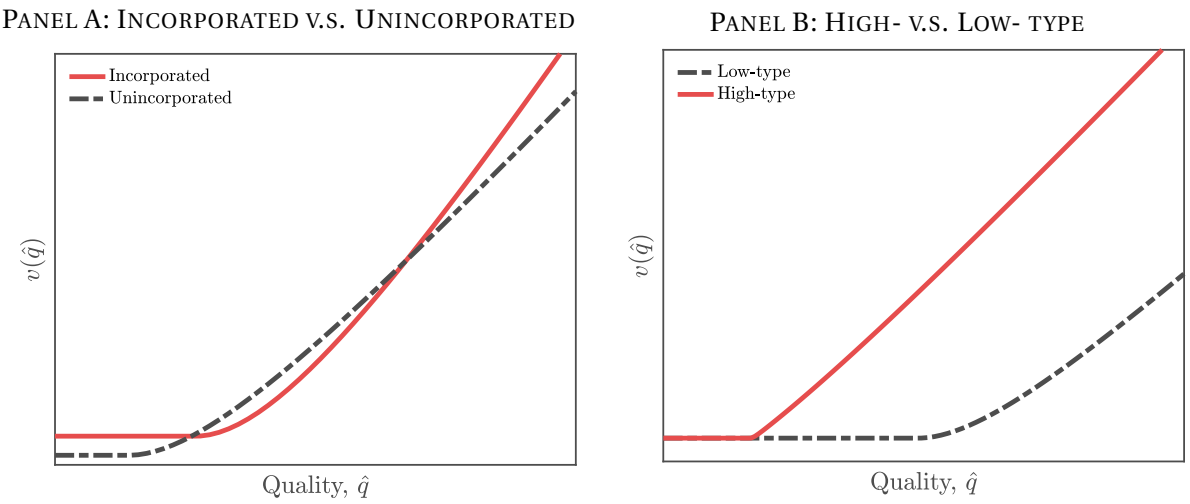
i.e. incentives to invest on quality depend on its marginal return of the investment $\frac{v_l(\hat{q}_+;\theta)-v_l(\hat{q};\theta)}{\hat{q}}$ as well as how efficient the investment technology is, θ. To provide further intuition regarding firm decision, Figures 1 and 2 depict a visual comparison of the value functions for different legal forms and firm types.

Panel A of Figure 1 shows the value of an incorporated and unincorporated firm by its size (quality) for a given firm type. First notice that firm value is constant below a certain quality threshold which determines the region of qualities where firms choose to exit. Since incorporated firm owners are protected by limited liability, their exit value is higher. Moreover, the value function of incorporated firms is steeper

than that of unincorporated firms when firm size is above a certain level. This means that marginal returns from improving quality under incorporation is higher. As a result, incorporated firms choose a higher expansion rate conditional on firm type, given by (13).

Panel B of Figure 1 makes the same comparison, but this time between a high-type and a low-type firm, given a legal form. The value function of high-type firms is steeper and the threshold quality at which they exit is lower. The former implies that they choose a higher expansion rate.[6] The latter indicates that high-types exit less often conditional on firm size. Therefore the presence of type heterogeneity generates exit rate heterogeneity conditional on firm size. This implication of the model is crucial in identifying the type heterogeneity, which will be discussed in Section 3.2.

Figure 1: FIRM VALUE



So far, I have discussed the expansion and exit decisions of firms based on Figure 1. Figure 2 focuses on legal form decision by showing how value differences between

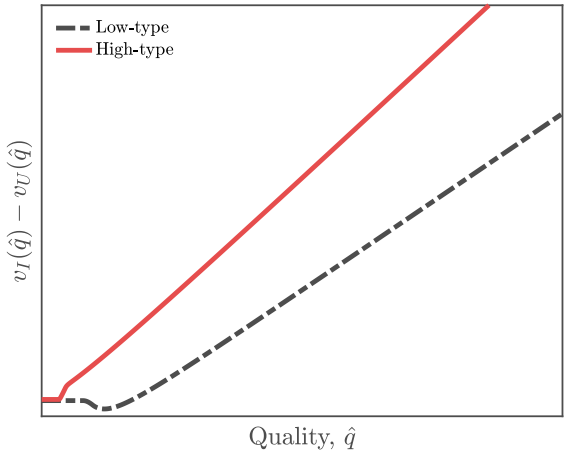
[6] In addition to the higher marginal returns from expansion, high-types also directly benefit from having a more efficient investment technology. That is, keeping the marginal returns constant, a higher θ implies a more efficient (cheaper) investment technology and implies a higher optimal expansion rate.

incorporated and unincorporated firms change by firm size for both high- and low-types. Although it shows some non-monotonicity for low-types, this difference typically increases in firm size. This implies that for a type θ firm, there exists a relative quality level $\hat{q}_\theta$ such that firms above this level of quality choose to incorporate:

$$v_I(\hat{q};\theta) - C_I \geq v_U(\hat{q},\theta), \quad for\ \hat{q} \geq \hat{q}_\theta. \tag{14}$$

Importantly, this threshold size is smaller for high-types ($\hat{q}_{\theta_H} < \hat{q}_{\theta_L}$) since the value difference between incorporated and unincorporated firms ($v_I(\hat{q};\theta) - v_U(\hat{q},\theta)$) is higher for high-types at any quality level. These results indicate that the likelihood of a firm being incorporated is increasing in firm size and higher for high-type firms conditional on firm size, highlighting the selection effects due to firm size and firm type heterogeneity.

Figure 2: CHOICE OF LEGAL FORM



Notes: This figure the differences between value of incorporated and unincorporated firms by size, conditional on firm type.

2.6 Firm Size Distribution and Aggregate Growth

As the quality improvements are stochastic in nature, firms (within each legal form and firm type category) are heterogeneous in terms of quality. Along balance growth path, the stationary distribution of relative qualities, which determines firm size, emerges as the result of the expansion and exit decisions of all firms, and characterizes the long-run state of the economy. For a given firm type θ, the distribution of relative qualities for incorporated firms in stationary equilibrium satisfies [7]

$$\begin{aligned} g\hat{q}f'_{I,\theta}(\hat{q}) &= (x_{I,\theta}(\hat{q}) + \kappa - g)f_{I,\theta}(\hat{q}) - x_{I,\theta}(\hat{q}_-)f_{I,\theta}(\hat{q}_-)\tfrac{1}{1+\lambda(1-\omega)} \\ &\quad -x_e\Psi(\hat{q}) - \mathbb{1}\left(\hat{q} \in \mathbb{Q}_{U\to I}\right)\mu f_{U,\theta}(\hat{q}) \end{aligned} \tag{15}$$

where $x_{I,\theta}(\hat{q})$ is the expansion rate, $f_{I,\theta}(\cdot)$ is the unnormalized density function with boundary conditions $f_{I,\theta}(\hat{q}) = 0$ for $\hat{q} < \hat{q}^{I,\theta}_{min}$ where $\hat{q}^{I,\theta}_{min}$ is the exit threshold quality solved from (11) and $\lim_{\hat{q}\to\infty} f_{I,\theta}(\hat{q}) = 0$. $\mathbb{Q}_{U\to I}$ denotes the quality region at which $v_I(\hat{q};\theta) - C_I \geq v_U(\hat{q},\theta)$ is satisfied, i.e. an unincorporated firm switches to incorporation. The distribution of qualities for unincorporated firms is analogous to above expression.

Given the distribution of firm qualities $f_{l,\theta}(.)$, the growth rate of average quality in the economy g is given by

$$g = \frac{\sum_{l\in\{I,U\}}\sum_{\theta\in\{\theta_L,\theta_H\}}\left[\int J(\hat{q},1)x_{l,\theta}(q)f_{l,\theta}(\hat{q})d\hat{q}\right] - \kappa + \frac{x_e}{\Phi}\mathbb{E}(\hat{q}_{entry})}{1 + \sum_{l\in\{I,U\}}\sum_{\theta\in\{\theta_L,\theta_H\}} f_{l,\theta}(\hat{q}^{l,\theta}_{min})\hat{q}^{l,\theta}_{min}} \tag{16}$$

where $J(\hat{q},1)$ is the amount of quality improvement defined in (7). The intuition for the growth rate in is as follows. The numerator has the contribution of entrants and different types and legal forms of incumbent firms to the quality distribution. This contribution happens at the rate $x_{l,\theta}(\hat{q})$, which underlines the connection between firm-level quality improvements and aggregate growth. The denominator on

[7] Details of the derivations are provided in Section 7.2 in Appendix.

the other hand adjusts for the improvements in quality distribution due to the firms exiting the economy endogenously.

2.7 Dynamic Equilibrium

Given the above description of the environment, I can now formally define the full dynamic equilibrium for this economy.

Definition 1 Consider the environment described above. A stationary equilibrium of this economy is a tuple

$$\{y_j, p_j, l_j, v_I(\hat{q};\theta), v_U(\hat{q};\theta), x_I(\hat{q};\theta), x_U(\hat{q};\theta), x_e, f_{I,\theta}(\hat{q}), f_{U,\theta}(\hat{q}), g\}$$

such that (i) representative household maximize utility; (ii) y_j and p_j maximize profits as in (4) and the labor demand l_j satisfies (3); (iii) $v_I(\hat{q};\theta)$ and $v_U(\hat{q};\theta)$ are given by the incorporated and unincorporated firm value functions in (11) and (12); (iv) $x_I(\hat{q};\theta)$ and $x_I(\hat{q};\theta)$ are given by the optimal expansion rate decision in (13) and x_e solves the entrants problem in (9); (v) the stationary equilibrium relative quality distributions $f_{I,\theta}(\hat{q})$ and $f_{U,\theta}(\hat{q})$ satisfy (15); (vi) the growth rate of average quality g is given by (16); (vii) labor market clears as in (20).

3. Data and Calibration Strategy

3.1 Data

The quantitative analysis of the model uses both firm-level and individual-level data for the years between 1999 and 2014. To measure the properties of the firm dynamics process, I rely on micro data for the population of non-farm and non-financial businesses from the Danish Business Statistics Register. The variables used in each year include the two-digit industry identifier, employment level, firm age, and legal

form of the business. As the focus of the paper is on how limited liability affects the incorporation decision and firm dynamics, I restrict the sample to firms with a single owner. This allows me to mitigate the importance of other incorporation benefits, such as issuing equity. To identify the owners of the businesses, I use the Danish Entrepreneurship Database which provides information on the primary founder of all privately owned firms in Denmark. I further restrict the sample to those firms that are active. I define active firms as firms with minimum employment of one full-time equivalent in addition to the founder. Following Gjerlv-Juel and Dahl (2012), I consider the firm exited after two successive years without activity.

The central moments in the calibration are firm entry rate and employment share of entrants in the economy, employment level by age, exit rate by age and size, the share of incorporated firms by age, transitions between the legal forms over time, and aggregate productivity growth. The moments related to entry/exit rates and legal form transitions are in per annum terms.

3.2 Calibration

I fix three of the parameters exogenously and calibrate the remaining parameters by minimizing the distance between several empirical moments and their model counterparts. Discount rate ρ, is set to 0.02, which roughly corresponds to an annual discount factor of 97%. The share of quality in final good β determines the price markup for firms through equation (4). Therefore I choose $\beta = 0.33$ to get a markup of 1.5 reported in De Loecker and Eeckhout (2018). The curvature of expansion production function η is set to 0.5, implying a quadratic expansion cost function, following Akcigit and Kerr (2017) and Acemoglu et al. (2018).

The remaining parameters, which are listed in Table 1, are calibrated by minimizing the distance between several empirical moments and their model counterparts. In particular, let Ω denote the set of parameters to be calibrated, M^E denote the vector of S empirical moments and $M(\Omega)$ denote the vector of model-simulated

moments. I then chose Ω to minimize the absolute relative deviation between the model and data:

$$\min_{\Omega} \sum_{m=1}^{S} \frac{|M_m^E - M_m(\Omega)|}{|M_m^E|}.$$

Even though the parameters are calibrated jointly, below I provide a heuristic description of the relationship between the parameters and the specific moments that are informative.

The expansion efficiencies for low-type and high-type firms and exit costs are mostly identified from the firms' employment growth and the growth differences between incorporated and unincorporated firms. Therefore, I use average employment growth from age 0 to age 20 for incorporated and unincorporated firms to discipline these parameters. I assume that entrants draw their initial quality from an exponential distribution, the rate parameter of which is identified from the employment share of entrants. On the other hand, the entry efficiency parameter θ_E is mainly determined by the aggregate entry rate.

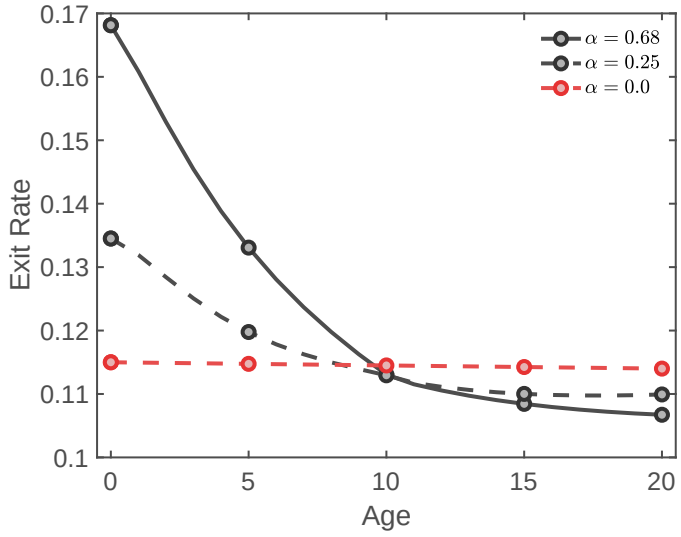
Since the fixed operation cost affects the threshold quality at which firms choose to exit endogenously, I use firm exit rates for incorporated and unincorporated firms to inform this parameter. The setup cost of incorporation is mainly identified by the share of entrepreneurs that choose to incorporate their business upon entry. Exit rates for large firms inform the exogenous death shock κ as it is the only cause for large firms to exit. As shown in Section 7.2 in Appendix, the model endogenously generates a Pareto-tailed distribution of firm size and the shape parameter of the distribution depends on ω which controls the extent of the spillovers in firm-level quality improvements. Therefore I target Pareto shape parameter implied by the empirical firm size distribution to pin down ω.[8] Aggregate growth rate is informative about the step size of quality improvements, λ.

To identify the share of low-type firms upon entry α, which determines the distribution of firm types among entrants, I focus on the age-profile of exit rates con-

[8] I use firms with more than 50 employees for the tail parameter estimation.

ditional on firm size. The model implies that without firm type heterogeneity in growth potential, the likelihood of exit would be independent of age *conditional on size* within a legal form. In data, however, such conditional exit rates are decreasing in firm age for both incorporated and unincorporated firms. Through the lens of the model, this pattern is rationalized by the interaction between firm heterogeneity and endogenous selection process: the share of high-type firms, which have lower exit rates conditional on size than low-type firms, increases within a given cohort as the cohort ages. This is shown in Figure 3 for incorporated firms, where I display the exit rate of small firms by age for different values of the share of low-types upon entry, α. Without any type heterogeneity, i.e. $\alpha = 0$, the conditional exit rates by age would be flat. Moreover, the lower the value of α, the less steep the decline in exit rate by age since the scope of selection is lower. Therefore, to inform α, I use the exit rate by age profile of small firms, the firms with less than or equal to 2 employees.

Figure 3: EXIT RATE BY AGE, CONDITIONAL ON SIZE (INCORPORATED FIRMS)



Notes: This figure shows the exit rate of small incorporated firms (firms from bottom 1% of the firm size distribution) by age for different values of share of low-type firms among entrants, α, while keeping the rest of the parameters constant.

4. Calibration Results

In this section I present the calibration results. Section 4.1 contains the structural parameters and targeted moments. In Section 4.2, I show that the calibrated model is also consistent with a variety of non-targeted moments.

4.1 Calibrated Parameters and Targeted Moments

Tables 1 and 2 contain the jointly calibrated parameters and the targeted moments, respectively. The estimates of the fixed cost of operation indicate that maintaining incorporated firms costs five times more than maintaining unincorporated firms. My estimates also show that high-type firms are about 2.5 times as efficient as the low-type firms in terms of improving their quality ($\theta_H/\theta_L \approx 2.5$) and entrants have a 68% chance of being a low-type firm ($\alpha = 0.678$). The parameter ω, which controls the weight of average quality in quality improvements, is estimated as 0.349, implying significant spillover effects between firms. The rate at which incumbent firms switch their legal form μ is estimated at 2.2%, reflecting the fact that legal form transitions among incumbent firms are rare.

Table 2 reports the targeted empirical moments and the predicted values from the model. It shows a good fit between model-implied moments and data. Overall, the model is able to replicate important characteristics of the data and the observed differences between incorporated and unincorporated firms. In particular, the model matches the better performance of incorporated firms in terms of employment growth: while incorporated firms grow by a factor of 4, compared to their entry size, by the time they are 20 years old; unincorporated firms reach only around 1.5 of their entry size. Moreover, the calibrated model also matches exit rate heterogeneity between incorporated and unincorporated firms in terms of both levels and changes by age: unincorporated firms have higher exit rates and the exit rates of small unincorporated firms show a steeper decline by age.

Table 1: PARAMETER ESTIMATES

Parameter		Value
Fixed cost (Incorporated)	ψ_I	0.250
Fixed cost (Unincorporated)	ψ_U	0.043
Exogenous death rate	κ	0.036
Exit cost	c_E	2.061
Step size	λ	0.156
Share of low types upon entry	α	0.678
Expansion efficiency (high type)	θ_H	0.932
Expansion efficiency (low type)	θ_L	0.392
Entry efficiency	θ_E	0.388
Incorporation setup cost	C_I	4.014
Entry dist. (rate)	χ	11.24
Legal form switching rate	μ	0.022
Share of average quality in step size	ω	0.349

4.2 Non-targeted Moments

In this section, I assess the performance of the calibrated model in how well it matches a variety of non-targeted moments. This strategy thus provides an out-of-sample test of the structure imposed by the model. Table 3 and Figure 4 summarize the results, which suggest that the model performs fairly well. In particular, the model is able to capture the average firm size differences between incorporated and unincorporated firms, the direction of legal form transition among incumbents, and the heterogeneity in firm size by age for both incorporated and unincorporated firms. Moreover, Figure 4 shows that the calibrated model performs well in replicating the share of incorporated firms in the overall economy as well as by firm size. This last result is especially reassuring as it suggests that the quantified model is able to capture the value of incorporation by firm size, which is reflected in the choice of incorporation by firms at different sizes.

Table 2: TARGETED MOMENTS

	Model	Data
Aggregate productivity growth	0.030	0.029
Entry rate	0.078	0.085
Employment share of entrants	0.031	0.022
Employment at age 20 (Incorporated)	4.14	4.28
Employment at age 20 (Unincorporated)	1.51	1.54
Share of incorporated firms at age 0	0.21	0.23
Share of incorporated firms at age 10	0.39	0.38
Exit rate (Incorporated)	0.08	0.06
Exit rate (Unincorporated)	0.17	0.12
Exit ratio of small firms, age 0 to 20 (I)	1.42	1.53
Exit ratio of small firms, age 0 to 20 (U)	2.46	2.67
Exit rate of large firms	0.036	0.036
Tail of firm size dist.	2.04	2.04
Share of incumbents switching legal form	0.046	0.043

Notes: Table reports both the data moments and the corresponding moments in the model. "Employment at age 20" moment refers to the average employment at age 20 relative to the entry employment level. I define small firms as firms with 1-2 employees in the data (including the firm owner).

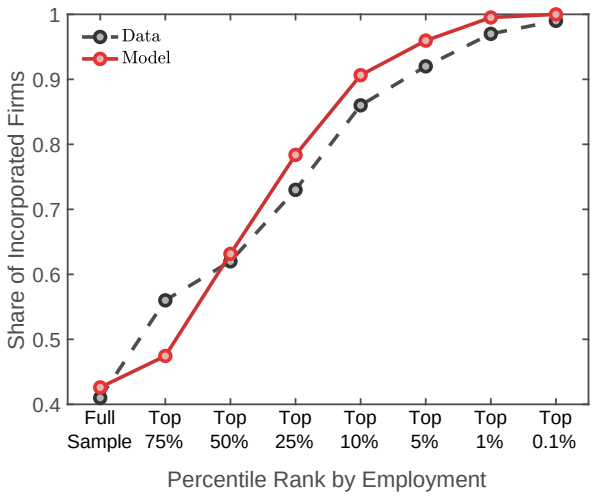
Table 3: NON-TARGETED MOMENTS

	Model	Data
Average firm size (I/U)	3.97	4.92
Share of switchers from U to I (cond. on switching)	0.93	0.99
Standard dev. of log employment age 10 (I)	1.49	1.32
Standard dev. of log employment age 10 (U)	1.12	0.95

5. Quantitative Results

In this section, I study the equilibrium properties of the calibrated model and its implications. I start by focusing on how the availability of incorporation choice affects firm incentives, equilibrium firm heterogeneity and the selection pattern in the economy. Then, to study the importance of the selection effect, I consider a counter-

Figure 4: SHARE OF INCORPORATION BY FIRM SIZE



Notes: This figure shows the share of incorporated firms in the top x% of the firm-size distribution for x = 0.1%, 1%, 5%, ... I report the data using a black dashed line and the model using a red solid line.

factual economy where the incorporation decision is randomized within firm types. Finally, I use the model to conduct two policy experiments to assess the value of incorporation.

5.1 Equilibrium Allocation: Firm Growth and Selection

Table 4 presents the key equilibrium objects for each legal form (incorporated or unincorporated) and firm type (low-type or high-type) category and it summarizes the heterogeneity in firms' growth incentives and in the composition of firm types within legal forms.

The first row reports the average expansion rates[9], the rate at which firms choose to improve their quality (see equation (13)). The expansion rate is a good summary statistics for firm growth as firms grow through quality improvements in the model. First, note that the ex-ante firm heterogeneity generates substantial firm growth rate

[9]Average expansion rate is calculated based on the firm size distribution within a legal form-firm type category.

differentials: conditional on legal form, the average expansion rate of high-type firms is around 7 times as high as that of low-type firms. Second, the choice of legal form also has direct effect on expansion rates and therefore firm growth: for both low- and high- types, incorporation increases average expansion rates by 50% (from 0.03 to 0.047 for low-types, from 0.20 to 0.31 for high-types), which can be considered as the *treatment effect* of incorporation on firm growth.[10] This treatment effect arises because, in the model, incorporation protects firm owners from downside risks, which incentivizes them to invest more in improving their product quality, subsequently grow large.

Table 4: FIRM GROWTH AND SELECTION

	Unincorporated		Incorporated	
	Low-type	High-type	Low-type	High-type
Average expansion rates	0.03	0.20	0.047	0.31
Shares among entrants	0.66	0.13	0.02	0.19
Shares among incumbents	0.36	0.22	0.005	0.42

Notes: The table contains various equilibrium objects for each legal form (incorporated or unincorporated) and firm type (low-type or high-type). The second and third rows refer to the share of each category such that they sum up to one. The model is parametrized according to Table 1.

The second and third row of Table 4 shows the distribution of firm types by legal form among entrants and incumbents such that the values in each row sum up to one. The former provides a measure for the selection of types into different legal forms upon entry, whereas the latter emphasizes the selection through competition, growth and the exit behavior of incumbent firms. These results reflect two important features of the entrants and incumbents.

First, consider the selection among entrants. Note that the *unconditional* probability of choosing incorporation upon entry is 21% (= 2% + 19%), however the probabilities *conditional* on firm type are drastically different: conditional on being a high-

[10] Note that this reflects the effect of incorporation on firm growth in partial equilibrium.

type, the probability of choosing incorporation upon entry is 59% $\left(= \frac{0.19}{0.13+0.19}\right)$. This result implies that high-type entrants disproportionately choose to become incorporated as high-types benefit more from incorporation compared to the low-types. This is because incorporation is valuable especially for large firms and high-type firms expect to grow larger than low-type firms. This pattern of firm types selecting into legal forms results in a significant firm type heterogeneity across legal forms: while the share of high-types is 90% among the incorporated entrants, this share among unincorporated entrants is only around 15%.

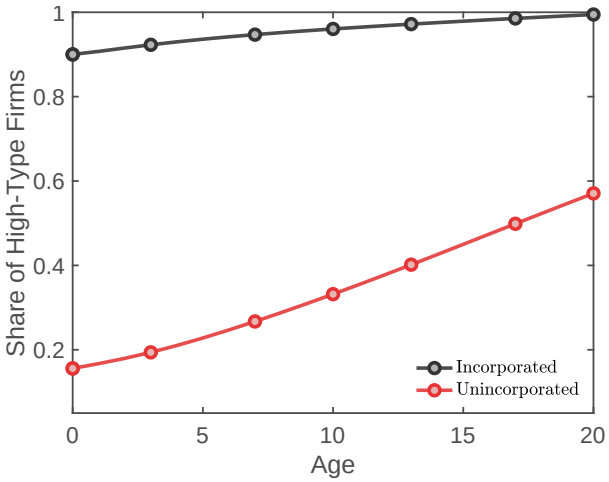
Second, the selection process becomes more pronounced among incumbents as is shown in the third row of Table 4. Note that the share of firms in a given legal form and firm type category would remain the same between entrants (row 2 in Table 4) and incumbents (row 3 in Table 4) if the average exit rates were uniform across these categories. In other words, the selection is driven by the resulting heterogeneity in the exit rates: the higher the exit rate differences, the stronger the selection effect. The results show that, within both incorporated and unincorporated incumbents, the share of high-types increases relative to entrants, showing a significant positive selection of high-types across the board and resulting in a 64% of high-type firm share in the economy. Importantly, the share of high-type incorporated firms shows the most significant increase compared to their entry share: the share of incorporated firms among incumbents reaches 43%, almost all of which are high-types.[11] Figure 5 also depicts the extent of selection among incumbents for a given cohort.

5.2 Counterfactual Exercise

To study the importance of the selection effect, I consider a counterfactual economy where the incorporation decision upon entry is randomized such that (i) the probabilities of choosing incorporation among low- and high- type entrants are same and

[11] Recall that share of incorporated firms among incumbents was not targeted in the calibration. Despite that, the model matches this moment very well as shown in Figure 4 (first data point from the left).

Figure 5: SHARE OF HIGH-TYPES BY AGE



Notes: The figure shows the share of high-type firms for a given cohort by ages and legal form for the baseline economy.

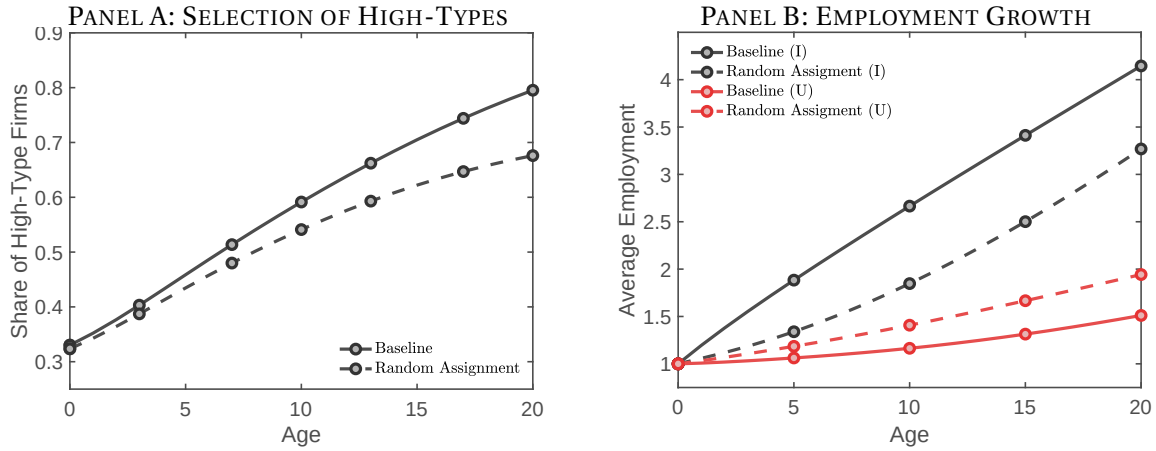
equal to the unconditional probability of incorporation in the baseline economy and (ii) the distribution of firm types upon entry are kept same as the baseline economy. This exercise effectively shuts down the selection effect among entrants.[12] Figure 6 illustrates the resulting effect of this counterfactual on selection pattern and firm growth. In Panel A, I depict the share of high-type firms of a given cohort by age. In both baseline and counterfactual economy, initial entrants have the same type heterogeneity by design. However as the cohort gets older, the share of high-types grows slower under the counterfactual economy, implying a weaker selection process. Overall, the share of high-types among incumbents decline from the baseline value of 64% to 57%. This is because, due to the randomization of legal form choice, a lower share of high-type firms benefits from incorporation (21% as opposed to 59% in the baseline). Therefore, on average, high-type firms grow slower and exit more often, compared to the baseline economy.

Panel B shows the effect of randomizing the legal form choice on employment

[12]Note that selection process among incumbents is still in place due to the heterogeneity in growth rates and exit decisions.

growth. As seen from the figure, the average employment by age differences between legal forms decrease significantly: for 20-year-old firms, the average employment difference between incorporated and unincorporated firms decreases by 44%. This pattern of the resulting change in employment growth leads to a decrease in the average size differences between legal forms by 32%. Overall, the aggregate productivity growth decreases from 3% to 2.7%.

Figure 6: COUNTERFACTUAL: RANDOM ASSIGNMENT OF LEGAL STATUS



Notes: Panel A depicts the share of high-type firms by age. Panel B shows the average employment by age for incorporated and unincorporated firms. I show the baseline model results using solid lines and the counterfactual model results using dashed lines.

5.3 Policy Experiments

I use the model to conduct two experiments to assess the value of incorporation. First, I consider a case where the incorporation option is eliminated, i.e. all firms are unincorporated. The effects of this experiment are summarized in Table 5. The absence of incorporation choice hurts the high-type firms the most as they are the ones that benefit most from the positive treatment effect of incorporation. The average expansion rate of high-type firms decreases significantly compared to the baseline economy. Low-types' expansion rate increases slightly, mainly because they ex-

perience less competitive pressure from high-types firms, incentivizing them to invest more in their quality improvements. High-type firms' lower incentive to grow in turn creates a weaker selection process in the economy. The share of high-type firms among incumbent is now lower (53% as oppose to 64% in the baseline economy).[13] The combination of lower expansion rates and weaker selection leads to a decline in aggregate productivity growth to 2.5% and welfare decreases by 4.6% (in consumption equivalent terms).

Table 5: ELIMINATING INCORPORATION CHOICE

	Low-type	High-type
Average expansion rates	0.04 (0.03)	0.22 (0.28)
Shares among entrants	0.68 (0.68)	0.32 (0.32)
Shares among incumbent	0.47 (0.36)	0.53 (0.64)

Notes: The table contains various equilibrium objects for an economy where incorporation choice is not available. The second and third rows refer to the share of each category such that they sum up to one. Numbers from the baseline model are given in parenthesis for comparison.

Next, I consider a policy that incentivizes incorporation by subsidizing the incorporated incumbent firms. In particular, I introduce a 5% subsidy to the incorporated incumbent firms' profit, which corresponds to 0.3% of the final output.[14] This policy not only encourages firms to choose incorporation but also incentivizes incorporated firms to invest more in quality improvements. Table 6 summarizes the impact of this policy on the equilibrium of the economy. As seen from Panel A, the subsidy policy increases the expansion rate of both low- and high- type incorporated firms but the increases are relatively small. The policy has a more significant impact on the selection pattern in the economy. The share of high-type firms that choose incorporation upon entry increases significantly to 75% (from 59% in the baseline) and half of the firms are high-type incorporated firms among incumbents. This subsidy

[13] Notice that share of firm types among entrance is the same as baseline economy by design.

[14] In order to focus on the implication of this policy on firm incentives and selection, I abstract from the costs of financing the subsidy.

policy would increase the aggregate productivity growth to 3.2%, mostly thanks to the change in the composition of firm types in favor of high-types who has higher expansion rates.

Table 6: SUBSIDIZING INCORPORATED FIRMS

	Unincorporated		Incorporated	
	Low-type	High-type	Low-type	High-type
	PANEL A: AVERAGE EXPANSION RATES			
Baseline	0.03	0.20	0.047	0.31
Subsidy	0.02	0.18	0.049	0.33
	PANEL B: SHARES AMONG ENTRANTS			
Baseline	0.66	0.13	0.02	0.19
Subsidy	0.63	0.09	0.01	0.27
	PANEL C: SHARES AMONG INCUMBENTS			
Baseline	0.36	0.22	0.005	0.42
Subsidy	0.32	0.16	0.011	0.51

Notes: The table contains various equilibrium objects for an economy where incorporated firm profit is subsidized by 5%, together with the baseline results for comparison. Panel B and C refer to the share of each category such that they sum up to one.

6. Conclusion

This paper develops an equilibrium model of firm dynamics with endogenous entry and exit, where firms spend resources to improve their productivity and choose whether to incorporate or not. I use the model to study how incorporation, which provides limited liability to firm owners, affects firm dynamics and macroeconomy. An important model feature is that firms have heterogeneous (high and low) types which differ in their capacity to improve productivity.

The model underlines two main effects that generate the differences in firm dynamics between incorporated and unincorporated firms. The first one is a treatment

effect of incorporation: since incorporation protects firms from downside risks, it incentivizes them to invest more in improving their productivity, subsequently grow large and exit less often. The second one is a selection effect due to the presence of firm heterogeneity: entrepreneurs with higher growth potential (i.e. more efficient in proving productivity) are more likely to choose incorporation as it is more valuable to large firms. The strength of this selection effect is determined by the interplay between endogenous entry, investment, and exit decisions.

To quantify the importance of these effects, I estimate the model with firm-level micro data from Denmark, specifically exploiting the heterogeneity in exit rates by age conditional on size to identify firm types in growth potential and therefore selection. My model fits the key moments from micro-data reasonably well, and also performs well on non-targeted moments in the data.

The calibration results suggest that accounting for firm heterogeneity in growth potential is quantitatively important in explaining the observed better performance of incorporated firms. In a counterfactual economy where the incorporation decision is randomized within firm types, both the productivity growth and the difference in the average size of incorporated and unincorporated firms would decline. To assess the value of incorporation, I also use the model to conduct two experiments. First, I consider a case where the option of incorporation is not available to the firms. The absence of incorporation not only eliminates the positive treatment effect on firms expansion rates but also mitigates the selection of high-growth potential firms in the economy, resulting in lower growth rates and welfare. Second, subsidizing the incorporated firms provides significant welfare gains. This is largely driven by the change in the degree of selection, i.e., the change in the composition of firm types.

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7. Appendix

7.1 Derivations for Static Product Market and Labour Market

By using the production function final goods sector, together with optimal price and quantity at the intermediate firm level, we can get a relation between wage rate and average quality:

$$Y = \frac{L^{\beta}}{1-\beta}\int_{\mathcal{N}} q_j \left(\left[(1-\beta)\frac{\bar{q}}{w}\right]^{\frac{1}{\beta}} L q_j\right)^{1-\beta} dj$$

$$Y = L(1-\beta)^{\frac{1-2\beta}{\beta}}\left[\frac{\bar{q}}{w}\right]^{\frac{1-\beta}{\beta}} \bar{q}\Phi \tag{17}$$

Also, final goods' producer profit needs to be zero (with aggregate price index normalized to 1). In other words, we have the following condition:

$$Y - \int_{\mathcal{N}} p_j k_j dj - wL = 0$$

$$Y = \frac{1}{(1-\beta)}\frac{w}{\bar{q}}\left[(1-\beta)\frac{\bar{q}}{w}\right]^{\frac{1}{\beta}} L\int_{\mathcal{N}} q_j dj + wL$$

$$Y = (1-\beta)^{\frac{1-\beta}{\beta}}\left[\frac{\bar{q}}{w}\right]^{\frac{1-\beta}{\beta}} L\bar{q}\Phi + wL \tag{18}$$

Using 17 and 18, we obtain

$$L(1-\beta)^{\frac{1-2\beta}{\beta}}\left[\frac{\bar{q}}{w}\right]^{\frac{1-\beta}{\beta}}\bar{q}\Phi = (1-\beta)^{\frac{1-\beta}{\beta}}\left[\frac{\bar{q}}{w}\right]^{\frac{1-\beta}{\beta}} L\bar{q}\Phi + wL$$

$$(1-\beta)^{\frac{1-2\beta}{\beta}}\left[\frac{\bar{q}}{w}\right]^{\frac{1-\beta}{\beta}}\bar{q}\Phi - (1-\beta)^{\frac{1-\beta}{\beta}}\left[\frac{\bar{q}}{w}\right]^{\frac{1-\beta}{\beta}}\bar{q}\Phi = w$$

$$\beta(1-\beta)^{\frac{1-2\beta}{\beta}}\Phi = \left[\frac{w}{\bar{q}}\right]^{\frac{1}{\beta}}$$

$$\frac{w}{\bar{q}} = \beta^{\beta}(1-\beta)^{1-2\beta}\Phi^{\beta}$$

$$w = \tilde{\beta}\bar{q} \tag{19}$$

where $\tilde{\beta} = \beta^{\beta}(1-\beta)^{1-2\beta}\Phi^{\beta}$. So wage is proportional to average quality in the economy. Incorporating the equilibrium wage rate, the profits simplify to

$$
\begin{aligned}
\pi(q_j) &= \beta\left[(1-\beta)\right]^{\frac{1-\beta}{\beta}}\left(\beta^{\beta}(1-\beta)^{1-2\beta}\Phi^{\beta}\right)^{\frac{\beta-1}{\beta}} Lq_j \\
\pi(q_j) &= \beta^{\beta}(1-\beta)^{(1-2\beta)\frac{\beta-1}{\beta}-\frac{1-\beta}{\beta}}\Phi^{\beta-1}Lq_j \\
\pi(q_j) &= \beta^{\beta}(1-\beta)^{2-2\beta}\Phi^{\beta-1}Lq_j \\
\pi(q_j) &= \frac{(1-\beta)}{\Phi}\beta^{\beta}(1-\beta)^{1-2\beta}\Phi^{\beta}Lq_j \\
\pi(q_j) &= \frac{(1-\beta)}{\Phi}\tilde{\beta}Lq_j \\
&= \frac{\beta^{\beta}(1-\beta)^{2-2\beta}}{\Phi^{1-\beta}}Lq_j
\end{aligned}
$$

This last expression makes it clear that the higher the firm mass, the lower the profits. This is because more firms imply higher wages, given the constant supply of workers, which reduces the profits. Furthermore, by combining 17 with 21, we can show that output is linear in $\bar{q}$

$$
\begin{aligned}
Y &= L(1-\beta)^{\frac{1-2\beta}{\beta}}\left[\beta^{\beta}(1-\beta)^{1-2\beta}\Phi^{\beta}\right]^{\frac{\beta-1}{\beta}}\bar{q}\Phi \\
&= (1-\beta)^{\frac{1-2\beta}{\beta}}\left[\beta^{\beta}(1-\beta)^{1-2\beta}\Phi^{\beta}\right]^{\frac{\beta-1}{\beta}}\Phi L\bar{q} \\
&= \beta^{\beta-1}(1-\beta)^{(\beta-1)\frac{1-2\beta}{\beta}+\frac{1-2\beta}{\beta}}\Phi^{\beta-1}\Phi L\bar{q} \\
&= \frac{(1-\beta)^{1-2\beta}}{\beta^{1-\beta}}\Phi^{\beta}L\bar{q}
\end{aligned}
$$

Finally, by combining 4 and 21, we can find L as

$$
\begin{aligned}
\tilde{L} &= \int_{\mathcal{N}} l_i dj \\
&= \int_{\mathcal{N}}\left[(1-\beta)\frac{\bar{q}}{w}\right]^{\frac{1}{\beta}} Lq_j\frac{1}{\bar{q}}dj \\
&= \left[(1-\beta)\frac{1}{\tilde{\beta}}\right]^{\frac{1}{\beta}} L\Phi.
\end{aligned}
$$

Labor market clearing implies that

$$\begin{aligned} 1 &= L + \tilde{L} \\ 1 &= L + \left[(1-\beta)\tfrac{1}{\beta}\right]^{\frac{1}{\beta}} L\Phi \\ L &= \tfrac{\beta}{\beta+(1-\beta)^2} \end{aligned} \tag{20}$$

Note that mass of firms does not affect L. If we substitute this to the profit, we get

$$\pi(q_j) = \frac{\beta^\beta(1-\beta)^{2-2\beta}}{\Phi^{1-\beta}} \frac{\beta}{\beta+(1-\beta)^2} q_j.$$

7.2 Quality Distributions

Denote q as relative qualities which follows

$$dq_t = -gqdt + J(q,\bar{q})dN_t$$

with poisson intensity x_q. Given this, the density of q at BGP satisfies the following Kolmogorov Forward Equation (KFE):

$$0 = (gqf)_q + x_{q-\eta} f(q-\eta) \times |1-\eta_q| - x_q f(q) + x_e \psi(q) - \kappa f(q)$$

where $\eta = \eta(q,\bar{q})$ is related to the inverse jump amplitude such that

$$q = \xi + J(\xi,\bar{q})$$

is the new state value corresponding to the old state value ξ, such that

$$\eta(q,\bar{q}) = J(\xi,\bar{q})$$

assuming J is monotonic in ξ so that J is invertible with respect to ξ, that the Jacobian

$$(1-\eta_q) = 1 - \frac{\partial \eta(q,\bar{q})}{\partial q}$$

is non-vanishing, and that the inverse transformation from ξ to q maps $(-\infty,+\infty)$ onto $(-\infty,+\infty)$. Let's consider a parametric form for J of the form

$$J(q,\bar{q}) = \lambda\left(\omega\bar{q} + (1-\omega)q\right), \quad \omega \in [0,1].$$

Given this form, the previous state is given by

$$q = \xi + \lambda\left(\omega\bar{q} + (1-\omega)\xi\right)$$
$$\xi = \frac{q - \lambda\omega\bar{q}}{1+\lambda(1-\omega)}$$

which gives

$$\begin{aligned}\eta(q,\bar{q}) &= \lambda\left(\omega\bar{q} + (1-\omega)\frac{q-\lambda\omega\bar{q}}{1+\lambda(1-\omega)}\right)\\ &= \frac{\lambda\omega\bar{q}}{1+\lambda(1-\omega)} + \frac{\lambda(1-\omega)q}{1+\lambda(1-\omega)}.\end{aligned}$$

Finally we have

$$\begin{aligned}|1-\eta_q(q,\bar{q})| &= 1 - \frac{\lambda(1-\omega)}{1+\lambda(1-\omega)}.\\ &= \frac{1}{1+\lambda(1-\omega)}.\end{aligned}$$

Therefore the density $f()$ is given by

$$0 = (gqf)_q + x_{q-\eta}f(q-\eta)\frac{1}{1+\lambda(1-\omega)} - x_q f(q) + x_e\psi(q) - \kappa f(q) \tag{21}$$

$$gqf_q = (x_q + \kappa - g)f(q) - x_{q-\eta}f(q-\eta)\frac{1}{1+\lambda(1-\omega)} - x_e\psi(q) \tag{22}$$

with $f(q) = 0$ for $q < q_{min}$ where q_{min} is solved from value function. Integrating over the domain $[q_{min}, \infty)$, we get

$$gq_{min}f(q_{min}) + \kappa\Phi = x_e \tag{23}$$

under the assumption that the density is integrable, i.e. $\lim_{q\to\infty} f(q) = 0$. Above equation simply implies that the amount of qualities going under q_{min} plus exits due to κ should be equal to the amount entering the system so that total mass is stable in stationary distribution.

Next let's look at the tail of the distribution, which will help us to solve the distribution. First note that as q goes to infinity, x_q becomes constant. We start with guessing that the distribution tail has a Pareto shape of the form $Cq^{-\zeta-1}$ as $q \to \infty$. Substituting this guess into the equation for the density delivers

$$-\zeta g C q^{-\zeta-1} + \bar{x}\left[C\left(\frac{q-\lambda\omega\bar{q}}{1+\lambda(1-\omega)}\right)^{-\zeta-1}\frac{1}{1+\lambda(1-\omega)} - Cq^{-\zeta-1}\right] + x_e\psi(q) - \kappa C q^{-\zeta-1} = 0$$

$$-\zeta g - \kappa + \bar{x}\left[q^{\zeta+1}\left(\frac{q-\lambda\omega\bar{q}}{1+\lambda(1-\omega)}\right)^{-\zeta-1}\frac{1}{1+\lambda(1-\omega)} - 1\right] + x_e\frac{\psi(q)}{Cq^{-\zeta-1}} = 0.$$

Now assume that entry distribution has a thinner tail, i.e.$\lim_{q\to\infty}\frac{\psi(q)}{Cq^{-\zeta-1}} = 0$. Then we have

$$\left[(1+\lambda(1-\omega))^{\zeta} - 1\right] = \frac{\zeta g + \kappa}{\bar{x}}. \tag{24}$$

Here one solution for ζ is zero which yields a degenerate solution. The next result partially characterize the non-degenerate solution.

Lemma 3 *The solution to ζ described in (24) is non-decreasing in ω and g and non-increasing in λ and τ for $\zeta \geq 1$. Moreover $\zeta = 1$ is a solution whenever $\lambda(1-\omega)\bar{x} = g+\kappa$ is satisfied. Finally $\lim_{\omega\to 1}\zeta(\omega) = \infty$.*

7.3 Derivation of Value Functions

The value function is given by

$$rV(q) - \frac{dV(q)}{dt} = \max\{0, \max_x\{\pi q - \eta\chi x^{\frac{1}{\eta}}q - c_F\bar{q} + x\left[V\left(q + \lambda\left(\omega\bar{q} + (1-\omega)q\right)\right) - V(q)\right] + \kappa(0 - V(q))\}$$

First let's define $V(q) = v(\hat{q})\bar{q}$ where $v(.)$ is the normalized value function and $\hat{q} = \frac{q}{\bar{q}}$. Then we divide both sides by $\bar{q}$ and get

$$\rho v(\hat{q}) - \frac{dv(\hat{q})}{dt} = \max\{0, \max_x\{\pi\hat{q} - \eta\chi x^{\frac{1}{\eta}}\hat{q} - c_F + x\left[v\left(\hat{q} + \lambda\left(\omega + (1-\omega)\hat{q}\right)\right) - v(\hat{q})\right] + \kappa(0 - v(\hat{q}))\}\},$$

where growth rate of $\bar{q}$ is g, $\frac{\dot{\bar{q}}}{\bar{q}} = g$ and we use the fact that $r = \rho + g$ from the representative consumer problem. Next we look at $\frac{dv(\hat{q})}{dt}$:

$$\begin{aligned}\frac{dv(\hat{q})}{dt} &= \frac{\partial v(\hat{q})}{\partial \hat{q}}\frac{\partial \hat{q}}{\partial \bar{q}}\frac{\partial \bar{q}}{\partial t} \\ &= \frac{\partial v(\hat{q})}{\partial \hat{q}} \times \left(-\frac{q}{\bar{q}^2}\right) \times \bar{q}g \\ &= \frac{\partial v(\hat{q})}{\partial \hat{q}} \times \left(-\frac{\hat{q}}{\bar{q}}\right) \times \bar{q}g \\ &= -g\hat{q}\frac{\partial v(\hat{q})}{\partial \hat{q}}.\end{aligned}$$

Therefore the final (stationary) value function is given by

$$\begin{aligned}\rho v(\hat{q}) = \max\{0, \max_x\{&\pi\hat{q} - \eta\chi x^{\frac{1}{\eta}}\hat{q} - c_F \\ &- g\hat{q}\frac{\partial v(\hat{q})}{\partial \hat{q}} \\ &+ x\left[v\left(\hat{q} + \lambda\left(\omega + (1-\omega)\hat{q}\right)\right) - v(\hat{q})\right] \\ &+ \kappa(0 - v(\hat{q}))\}\}.\end{aligned}$$

The above value function is for the incorporated firms. For unincorporated firms, when firm exit with κ arrival rate, the value goes to a negative value, instead of zero. For this, if we assume that this is proportional quality, i.e.,

$$rV(q) - \frac{dV(q)}{dt} = \max\{0, \max_x\{\pi q - \eta\chi x^{\frac{1}{\eta}}q - c_F\bar{q} + x\left[V\left(q + \lambda\left(\omega\bar{q} + (1-\omega)q\right)\right) - V(q)\right] + \kappa(-c_E \times q - V(q))\}$$

This will result in the following normalized value function

$$\begin{aligned}\rho v(\hat{q}) = \max\{0, \max_x\{&(\pi - \kappa c_E)\hat{q} - \eta\chi x^{\frac{1}{\eta}}\hat{q} - \psi_F \\ &- g\hat{q}\frac{\partial v(\hat{q})}{\partial \hat{q}} \\ &+ x\left[v\left(\hat{q} + \lambda\left(\omega + (1-\omega)\hat{q}\right)\right) - v(\hat{q})\right] \\ &+ \kappa(0 - v(\hat{q}))\}\}.\end{aligned}$$

which is similar to a proportional decrease in the per period profits.

7.4 Growth Rate

Let $Q_t \equiv \int_{\mathcal{N}} q_j(t)dj$ be the sum of qualities in the economy, which is the relevant measure for aggregate growth rate. We can express Q_t after an instant Δt as

$$\begin{aligned} Q_{t+\Delta t} = Q_t &+ \int x_q \Delta t \lambda \left[\omega \bar{q}_t + (1-\omega) q_t\right] f(q_t) dq_t \\ &- \kappa \Delta t Q_t \\ &- g q_{min,t} \Delta t f(q_{min,t}) \\ &+ x_e \Delta t \mathbb{E}(q_t^{entry}) \end{aligned}$$

where $f(.)$ is the density function for quality distribution. With this, we can derive the growth rate as

$$\begin{aligned} Q_{t+\Delta t} &= Q_t + \int x_q \Delta t \lambda \left[\omega \bar{q}_t + (1-\omega) q_t\right] f(q_t) dq_t - \kappa \Delta t Q_t - g q_{min,t} \Delta t f(q_{min,t}) + x_e \Delta t \mathbb{E}(q_t^{entry}) \\ &= Q_t + \Delta t \lambda \omega \frac{Q_t}{\Phi} \Phi \mathbb{E}(x_q) \\ &\quad + \int x_q \Delta t \lambda (1-\omega) q_t f(q_t) dq_t - \kappa \Delta t Q_t - g q_{min,t} \Delta t f(q_{min,t}) + x_e \Delta t \mathbb{E}(q_t^{entry}) \\ g \equiv \frac{Q_{t+\Delta t} - Q_t}{\Delta t Q_t} &= \lambda \omega \mathbb{E}(x_q) + \lambda(1-\omega) \frac{\int x_q q_t f(q_t) dq_t}{Q_t} - \kappa - g f(q_{min,t}) \frac{q_{min,t}}{Q_t} + x_e \frac{\mathbb{E}(q_t^{entry})}{Q_t} \\ &= \frac{\lambda \omega \mathbb{E}(x_q) + \lambda(1-\omega) \frac{\int x_q q_t f(q_t) dq_t}{Q_t} - \kappa + x_e \frac{\mathbb{E}(q_t^{entry})}{Q_t}}{1 + f(q_{min,t}) \frac{q_{min,t}}{Q_t}} \end{aligned}$$

Note that we solve the distribution for the relative qualities. Suppose we normalize the qualities with average quality $\bar{q}_t \equiv \frac{Q_t}{\Phi}$, then the above growth expression can be written as

$$\begin{aligned} g &= \frac{\lambda \omega \mathbb{E}(x_q) + \lambda(1-\omega) \frac{\int x_q q_t f(q_t) dq_t}{\bar{q}_t \Phi} - \kappa + x_e \frac{\mathbb{E}(q_t^{entry})}{\bar{q}_t \Phi}}{1 + f(q_{min,t}) \frac{q_{min,t}}{\bar{q}_t \Phi}} \\ &= \frac{\lambda \omega \mathbb{E}(x_q) + \lambda(1-\omega) \frac{\int x_q \hat{q} f(\hat{q}) d\hat{q}}{\Phi} - \kappa + x_e \frac{\mathbb{E}(q_t^{entry})}{\bar{q}_t \Phi}}{1 + f(\hat{q}_{min}) \frac{\hat{q}_{min}}{\Phi}} \\ &= \frac{\lambda \omega \mathbb{E}(x_q) + \lambda(1-\omega) \mathbb{E}(x_q \hat{q}) - \kappa + x_e \frac{\mathbb{E}(q_t^{entry})}{\bar{q}_t \Phi}}{1 + f(\hat{q}_{min}) \frac{\hat{q}_{min}}{\Phi}} \end{aligned}$$

Notice that, in this case, $\mathbb{E}(\hat{q}) = 1$. This is convenient because it allows $\bar{\hat{q}}$ in step size to be 1.

7.5 Boundary Behavior of Firm Value

We can show that value function is linear in q as q goes to infinity when profits are is linear in q. First notice that as $q \to \infty$, $\bar{q}$ and fixed cost of operation becomes insignificant, therefore

$$\lim_{q\to\infty} q + J(q, \bar{q}) = q \times (1 + \lambda(1 - \omega))$$

When ω is equal to one, there is no benefit of innovating as q gets very large. Now lets guess that the value function is of the form $v = Cq$. By substituting below

$$\rho Cq = \max_x \left\{ \phi q - \frac{\chi}{2} q x^2 - gqC + xC\left[q \times (1 + \lambda(1 - \omega)) - q\right] \right\}$$

$$\begin{aligned} \chi q x_* &= Cq\lambda(1 - \omega) \\ x_* &= \frac{C\lambda(1 - \omega)}{\chi} \end{aligned}$$

which is a constant. By substituting this above, we get

$$\begin{aligned} \rho Cq &= \phi q - \frac{\chi}{2} q x_*^2 - gqC + x_* C\left[q \times (1 + \lambda(1 - \omega)) - q\right] \\ \rho C &= \phi - \frac{\chi}{2} x_*^2 - gC + x_* C\lambda(1 - \omega) \\ \rho C &= \phi - \frac{1}{2\chi}\left(C\lambda(1 - \omega)\right)^2 - gC + \frac{\left[C\lambda(1 - \omega)\right]^2}{\chi} \\ 0 &= C^2 \frac{\left(\lambda(1 - \omega)\right)^2}{2\chi} - (\rho + g)C + \phi \end{aligned}$$

which solves the constant. The roots are

$$C_{-+} = \frac{\rho + g \pm \sqrt{(\rho + g)^2 - 2\frac{\left(\lambda(1-\omega)\right)^2}{\chi}\phi}}{\frac{\left(\lambda(1-\omega)\right)^2}{\chi}}$$

Note that C_+ is never a solution we are looking for because it makes net profit negative. Therefore the slope of the value function is given by C_-.